Non-degenerate Diophantine equations

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Outline

This talk is on joint work (arXiv:2501.17155) with Shijie Fan. I will focus on explicit definitions and simple special cases of our results.

Program:

1. Two classical examples: Clebsch–Klein equations
2. More explicit examples: Ico equations
3. Application: Fermat problem
4. Non-degeneracy: Definition and basic idea
5. Non-degenerate curves
6. Equations $F(x, y) = 0$

1. Two classical examples: The Clebsch–Klein equations

Write $\mathbb{N} = \mathbb{Z}_{\geq 1}$, for $i \in \mathbb{N}$ let σ_i be the i-th elementary symmetric polynomial

Icosahedron surface (Klein, 1873): $V \subset \mathbb{P}^4_{\mathbb{Z}} : \sigma_2 = 0 = \sigma_4$

Cubic diagonal surface (Clebsch, 1871): $V' \subset \mathbb{P}^4_{\mathbb{Z}} : z_0^3 + \ldots + z_4^3 = 0 = \sigma_1$

Geometry. After Clebsch and Klein, many others studied geometric aspects of the complex surfaces $V_{\mathbb{C}}$, $V'_{\mathbb{C}}$ which are both rational (i.e. birational to $\mathbb{P}^2_{\mathbb{C}}$).

Non-degenerate equations. Geometric results of Hirzebruch (1976) imply that the defining equations of the surfaces $V_{\mathbb{Q}}$ and $V'_{\mathbb{Q}}$, given respectively by

$$\sigma_2 = 0 = \sigma_4 \quad \text{and} \quad z_0^3 + \ldots + z_4^3 = 0 = \sigma_1,$$

are two classical examples of equations which we call non-degenerate.

Clebsch–Klein surfaces: Diophantine equations

Fundamental Diophantine equations can be studied via the surfaces

$$V \subset \mathbb{P}^4_{\mathbb{Z}} : \sigma_2 = 0 = \sigma_4 \quad \text{and} \quad V' \subset \mathbb{P}^4_{\mathbb{Z}} : z_0^3 + \ldots + z_4^3 = 0 = \sigma_1.$$

Let S be a finite set of rational primes, $N_S = \prod_{p \in S} p$, $\mathbb{Z}_S = \mathbb{Z}[\frac{1}{N_S}]$.

Equation 1. $U(\mathbb{Z}_S) = \{x \in \Sigma;\ \sigma_2(x) = 0 = \sigma_4(x)\}$ where

$$\Sigma = \{x \in \mathbb{Z}^5;\ \gcd(x_0, \ldots, x_4) = 1 \text{ and } \gcd(x_i, x_j, x_k) \in \mathbb{Z}_S^\times \text{ if } i < j < k\}$$

Equation 2. $U'(\mathbb{Z}_S) = \{x \in (\mathbb{Z}_S^\times)^4;\ x_1^3 + \ldots + x_4^3 = 1 = \sigma_1(x)\}$

Relation to V, V'. A direct computation shows that $U(\mathbb{Z}_S)$ and $U'(\mathbb{Z}_S)$ identify respectively with the sets of S-integral points of

$$U = V \setminus Z,\ Z = \cup_{i=0}^4 P_i \quad \text{and} \quad U' = V' \setminus D,\ D = \cup_{i=0}^4 V_+(z_i).$$

Here P_i are the (images of the) five $\mathbb{Z}$-points of V obtained by permuting the coordinates of $(1, 0, \ldots, 0)$, and $V_+(z_i)$ are the five coordinate hyperplanes.

Comparing U, U'. Equation 1 in $U(\mathbb{Z}_S)$ is more fundamental than Equation 2 in $U'(\mathbb{Z}_S)$ since U is much larger than U': Over $\mathbb{Z}[\frac{1}{3}]$ we get an open immersion

$$\varphi : U' \hookrightarrow U, \qquad U' \cong^{\varphi} V \setminus D \subsetneq U = V \setminus Z,$$

with $\frac{1}{4} h(x) \leq h(\varphi(x)) \leq 4h(x)$ for all $x \in U'(\mathbb{Q})$ and h is usual log Weil height.

Some Diophantine results

The infinite sets $U'(\mathbb{Q}) \hookrightarrow U(\mathbb{Q})$ are both very large. A highly uniform bound for the number of solutions of the generalized unit equation implies:

Theorem (1) (special case of Evertse–Schlickewei–Schmidt, 2002)

It holds $|U'(\mathbb{Z}_S)| = |\{x \in (\mathbb{Z}_S^\times)^4; \sum x_i^3 = 1 = \sigma_1(x)\}| \leq \exp(24^{12}(4|S|+1))$.

What about $W'(\mathbb{Z}_S)$ when $U' \subsetneq W' \subsetneq V'$? Very strong (essentially optimal) non-degeneration results for S-integral points on smooth cubic surfaces imply:

Theorem (2) (special case of Corvaja–Zannier, 2010)

The points $W'(\mathbb{Z}_S)$ *are not Zariski dense if* $W' = V' \setminus \big(V_+(z_0) \cup V_+(z_1)\big)$.

What about $U(\mathbb{Z}_S) = \{x \in \Sigma, \sigma_2(x) = 0 = \sigma_4(x)\}$? If h is the usual logarithmic Weil height and $c = 10^\kappa$ for $\kappa = 10^{12}$, then explicit bounds for the height and the number of S-integral points on coarse Hilbert moduli schemes imply:

Theorem (3) (special case of K.–Kret, 2023)

It holds $|U(\mathbb{Z}_S)| \leq (cN_S)^\kappa$ *and all* $x \in U(\mathbb{Z}_S)$ *satisfy* $h(x) \leq cN_S^{24}$.

Theorem 3 holds also for U' in place of U using $\varphi : U' \hookrightarrow U = V \setminus (\cup_{i=0}^4 P_i)$, and Theorem 3 is optimal in the sense that one can not make $U \subset V$ larger.

Some Diophantine results: Discussion

Theorem (1) (special case of Evertse–Schlickewei–Schmidt, 2002)

It holds $|U'(\mathbb{Z}_S)| = |\{x \in (\mathbb{Z}_S^\times)^4; \sum x_i^3 = 1 = \sigma_1(x)\}| \leq \exp(24^{12}(4|S|+1))$.

Theorem (2) (special case of Corvaja–Zannier, 2010)

The points $W'(\mathbb{Z}_S)$ *are not Zariski dense if* $W' = V' \setminus \big(V_+(z_0) \cup V_+(z_1)\big)$.

Theorem (3) (special case of K.–Kret, 2023)

It holds $|U(\mathbb{Z}_S)| \leq (cN_S)^\kappa$ *and all* $x \in U(\mathbb{Z}_S)$ *satisfy* $h(x) \leq cN_S^{24}$.

- Theorems 1 and 2 hold for any number field K, while Theorem 3 only holds for $K = \mathbb{Q}$ and so far its proof does not generalize to any K.
- So far the techniques underlying Theorem 3 do not allow to study the much more difficult $W'(\mathbb{Z}_S)$, and the bound for $|U'(\mathbb{Z}_S)|$ provided by Theorem 3 and $|U'(\mathbb{Z}_S)| \leq |U(\mathbb{Z}_{S\cup\{3\}})|$ is much worse than Theorem 1.
- Theorems 1 and 2 use certain Diophantine approximations techniques which allow to prove deep Diophantine properties of the solutions, but so far they do not allow to explicitly bound the height of all solutions.
- So far is not clear whether Diophantine approximation techniques can be applied to study $U(\mathbb{Z}_S)$ since the relevant divisors in the desingularization of the normal surface $V_{\mathbb{Q}}$ are never big (and thus never ample).

2. More explicit examples: lco equations

Ico equations. Let $f \in \mathbb{Z}[x_0, \ldots, x_4]$ be homogeneous of degree $n \geq 1$:

- $X_f \subset \mathbb{P}^4_{\mathbb{Z}} : \sigma_2 = 0 = \sigma_4, f = 0$ for σ_i the i-th elementary symmetric
- $X_f(\mathbb{Q}) = \{x \in \mathbb{Z}^5; \sigma_2(x) = 0 = \sigma_4(x), f(x) = 0, \gcd(x_0, \ldots, x_4) = 1\}$

Non-degenerate criterion:

- $\Delta = \prod a_i$ with the product taken over the five diagonal coefficients $a_0, \ldots, a_4$ of the homogeneous $f = \sum a_i x_i^n + \ldots$ of degree n
- If $\Delta \neq 0$ then we call the curve X_f non-degenerate.

Non-degenerate equations. The equations $(\sigma_2 = 0 = \sigma_4, f = 0)$ with f having $\Delta \neq 0$ are all explicit examples of non-degenerate equations.

Theorem (A)

If $\Delta \neq 0$ then $|X_f(\mathbb{Q})| \leq c \cdot \mathrm{rad}(\Delta)^{\kappa}$ and $h(x) \leq c \cdot \mathrm{rad}(\Delta)^{24}$ for all $x \in X_f(\mathbb{Q})$.

Here $h(x) = \log \max |x_i|$, and one can take for example $c = 10^{\kappa}$ and $\kappa = 10^{12}$

The locus $\Delta \neq 0$ is open inside moduli and almost all X_f are non-degenerate

Application. The curves X_f are fundamental for the study of rational points on curves since any curve X over $\mathbb{Q}$ of genus ≥ 2 is birational over $\mathbb{Q}$ into some X_f. As almost all X_f are non-degenerate, Theorem A allows to establish the effective Mordell conjecture for large classes of (explicit) curves X over $\mathbb{Q}$.

3. Application: Fermat problem

Frow now on we write $\mathbb{P}^m$ for $\mathbb{P}^m_{\mathbb{Q}}$. Let $f_1, f_2 \in \mathbb{Z}[x_0, \ldots, x_4]$ be homogeneous such that $S \subseteq \mathbb{P}^4 : f_1 = 0 = f_2$ is a projective rational surface over $\mathbb{Q}$ and define

$$S(\mathbb{Z}) = \{x \in \mathbb{Z}^5;\ f_1(x) = 0 = f_2(x) \text{ and } \gcd(x_0, \ldots, x_4) = 1\}.$$

Call $x \in S(\mathbb{Z})$ trivial if all $x_i \in \{-1, 0, 1\}$ and write $\mathbb{N} = \mathbb{Z}_{\geq 1}$.

Problem (F). *For any $a, b, c, d, e \in \mathbb{Z} - \{0\}$, try to construct n_0 in $\mathbb{N}$ such that all solutions of the Fermat equations (F_n) are trivial when $n \geq n_0$:*

$$(F_n) \quad ax_0^n + bx_1^n + cx_2^n + dx_3^n + ex_4^n = 0, \quad x \in S(\mathbb{Z}), \quad n \in \mathbb{N}.$$

As S is a rational surface over $\mathbb{Q}$, the infinite set $S(\mathbb{Q})$ is 'large' and hence the Diophantine problem (F) is 'non-trivial'. In particular, for any $n \in \mathbb{N}$ there exist nonzero $a, b, c, d, e \in \mathbb{Z}$ such that (F_n) has a non-trivial solution and thus n_0 has to depend on a, b, c, d, e if it exists. We conjecture that (F) can be solved if and only if $(S \cap Z)(\mathbb{Q})$ is trivial, where $Z \subset \mathbb{P}^4$ is given by

$$Z = \cap_j \cup_{i \neq j} V_+(f_{ij}), \quad f_{ij} = x_i^2 x_j - x_j^3. \tag{1}$$

If $S = \mathbb{P}^2 \subset \mathbb{P}^4 : x_3 = 0 = x_4$, then the Fermat problem (F) is the classical Fermat problem solved by Wiles for $a = b = -c = 1$ with optimal $n_0 = 3$.

Now, we replace the surface $S = \mathbb{P}^2$ by the birationally equivalent surface $S^{\text{ico}} \subset \mathbb{P}^4 : \sigma_2 = 0 = \sigma_4$. Then our Theorem (A) allows us to solve the Fermat problem (F) for $S = S^{\text{ico}}$.

Corollary. *If $S = S^{\text{ico}}$ then all solutions x of (F_n), $n \in \mathbb{N}$, satisfy*

$$\log \max |x_i| \leq k \cdot \nu^{\kappa}, \quad \nu = \text{rad}(abcde), \quad k = 10^{10^{12}}, \quad \kappa = 24.$$

Moreover there is $n_0 \in \mathbb{N}$ such that all solutions of (F_n) are trivial when $n \geq n_0$.

The analogue of this Corollary is still open for general a, b, c in the classical case $S = \mathbb{P}^2$. However if $S = \mathbb{P}^2$ then for large classes of a, b, c optimal results are known and many of these results actually hold in more general situations (e.g. with different exponents n_1, n_2, n_3 or in certain number fields $K \neq \mathbb{Q}$).

As the underlying geometry of (F) is equivalent, we conjecture that (F) behaves similarly (or might be even related) for $\mathbb{P}^2$, S^{ico} and other rational surfaces S.

4. Non-degeneracy: Definition and basic idea

Let $m, n \in \mathbb{Z}_{\geq 1}$, $i \in \{1, \ldots, m\}$, $f_i \in \mathbb{Z}[x_0, \ldots, x_n]$ homogeneous of degree ≥ 1.

Definition. The equations $(f_1 = 0, \ldots, f_m = 0)$ are called non-degenerate if $V(f_1, \ldots, f_m) \subseteq \mathbb{P}^n$ satisfies a certain geometric non-degenerate criterion.

Key property. If V is non-degenerate then there exists an open dense $U \subseteq V$ which is a coarse moduli scheme of finite level: Roughly speaking this means

$$U(\bar{\mathbb{Q}}) \cong \{(A, \alpha);\ A \in \underline{A}_g, \alpha \in \mathcal{P}(A)\}, \quad \underline{A}_g = \{A/\bar{\mathbb{Q}} \text{ AV of } \dim(A) = g\}$$

for $g \in \mathbb{Z}_{\geq 1}$ and $\mathcal{P}(A)$ a finite set associated to the AV=abelian variety $A/\bar{\mathbb{Q}}$.

Idea. Study subset $\mathcal{S}$ of $\{x \in \mathbb{Z}^{n+1}; f_i(x) = 0, \gcd(x_j) = 1\} = V(\mathbb{Q})$ via

$$V(\mathbb{Q}) = U(\mathbb{Q}) \cup (V \setminus U)(\mathbb{Q}) \quad \text{and} \quad U(\mathbb{Q}) = \phi^{-1}(\phi(U(\mathbb{Q}))$$

where $\phi : U(\mathbb{Q}) \to \underline{A}_g$ is simply the forgetful map

$$\phi : U(\mathbb{Q}) \subset U(\bar{\mathbb{Q}}) \to \underline{A}_g \quad (A, \alpha) \mapsto A.$$

Here $(V \setminus U)(\mathbb{Q})$ is usually simpler than $V(\mathbb{Q})$ since $\dim(V \setminus U) < \dim(V)$; e.g. $V \setminus U$ is automatically finite if $m = 1, n = 2$ (i.e. $V = V_+(f_1) \subset \mathbb{P}^2$ is a curve).

Strategy to prove explicit height bounds

Suppose $(f_1 = 0, \ldots, f_m = 0)$ are non-degenerate, $V = V(f_i)$, $h = \log \max |x_j|$

Goal: For $\mathcal{S} \subset \{x \in \mathbb{Z}^{n+1}; f_i(x) = 0, \gcd(x_j) = 1\} = V(\mathbb{Q})$, try to bound h on $\mathcal{S}$

We use a strategy which was developed over the last 70 years by many people: It combines the method of Faltings (Arakelov, Parsin, Szpiro) with modularity and Masser–Wüstholz isogeny estimates. Very rough outline:

(a) Effective Parsin. Try to control h on $(V \setminus U)(\mathbb{Q})$ and try to show that the forgetful map $\phi : U(\bar{\mathbb{Q}}) \to \underline{A}_g$ satisfies $h(x) \ll h_F(\phi(x))$ for all $x \in U(\bar{\mathbb{Q}})$, where h_F is the stable Faltings height on $\underline{A}_g$.

(b) Effective Shafarevich. Try to explicitly bound h_F on $\mathcal{A} = \phi\big(\mathcal{S} \cap U(\mathbb{Q})\big)$:

(i) To control the variation of h_F in the isogeny class of each $A \in \mathcal{A}$, combine a formula of Faltings with the deep Masser–Wüstholz isogeny estimates based on transcendence (fully explicit version of Gaudron–Rémond).

(ii) Try to control h_F on $\mathcal{A}$ modulo isogenies, by combining geometric version of modularity (using Tate conjecture proven by Faltings) with averaged Colmez formula for A with CM, explicit analytic estimates and explicit results from Arakelov theory. This part (ii) works unconditionally in the GL_2-case using Serre's modularity conjecture (proven by Khare and Wintenberger), GL_2-type constructions, Belyi degree results, ...

Remark. The effective Parsin step (a) is usually a purely geometric/analytic problem, while step (b) usually requires deep arithmetic results.

Suppose $(f_1 = 0, \ldots, f_m = 0)$ are non-degenerate. Then $V = (f_1, \ldots, f_m) \subseteq \mathbb{P}^n$ contains an open dense $U \subseteq V$ which is a coarse moduli scheme of finite level.

The strategy currently can be applied in the following situations:

(GL_2) **Thm (K., 2013).** The variety $U/\mathbb{Q}$ is a moduli scheme of GL_2-type: All points of U parametrize (A, α) where A is an AV of GL_2-type.

(H) **Thm (K.–Kret, 2019).** The variety $U/\mathbb{Q}$ is a Hilbert moduli scheme: All points of U parametrize (A, α) where A is an AV with real multiplications.

(cH) **Thm (K.–Kret, 2023).** The variety $U/\mathbb{Q}$ is a coarse Hilbert moduli scheme with empty branch locus: $U(\bar{\mathbb{Q}})$ parametrizes (A, α) as in (H) and the underlying stack $\mathcal{M}_{\mathcal{P}}$ of (A, α) is étale over U via $\pi : \mathcal{M}_{\mathcal{P}} \to U$.

Strategy to prove explicit height bounds: Examples

The strategy currently can be applied in the following situations:

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Examples.

- If V, V' are the Clebsch–Klein surfaces of part 1, then U, U' satisfy (cH).
- If X_f in part 2 is non-degenerate, then $V = U = X_f$ satisfies (cH).
- All curves $V \subset \mathbb{P}^2_{\mathbb{Q}}$ satisfying Criterion (τ) in part 6.

The following examples still require to work out the effective Parsin step (a):

- If there exists an abelian scheme A_0/U of GL_2-type, then U satisfies (GL_2).
- If U admits a quasi-finite morphism to a representable Hilbert modular variety over $\mathbb{Q}$, then U satisfies (H).
- Alpöge's examples (including a nice explicit family) in his thesis in which he studied the problem of constructing curves $V = U$ satisfying (GL_2) or (H).

5. Non-degenerate curves

Write $\mathbb{A}^n = \mathbb{A}^n_{\mathbb{Q}}$ for $n \in \mathbb{N}$

Let X be a curve[1] over $\mathbb{Q}$. For example if $F \in \mathbb{Q}[x, y]$ has degree ≥ 1 then $X = V(F) \subset \mathbb{A}^2$ is a curve over $\mathbb{Q}$ with $X(\mathbb{Q}) = \{(x, y) \in \mathbb{Q}^2; F(x, y) = 0\}$.

Classical result. Each irreducible component of X is birational[2] over $\mathbb{Q}$ into some plane curve $V(F) \subset \mathbb{A}^2$.

On using classical constructions of Clebsch and Klein, we obtain analogue:

Theorem (B)

Each irreducible component of X is birational over $\mathbb{Q}$ into some curve X_f.

Definitions.

- If X is birational over $\mathbb{Q}$ into some curve $V(F) \subset \mathbb{A}^2$ then call $V(F)$ plane model of X.
- If X is birational over $\mathbb{Q}$ into some curve X_f then call X_f ico model of X.

[1] A curve over a field k is a finite type separated k-scheme whose irreducible components all have dimension one.

[2] A curve X over $\mathbb{Q}$ is birational over $\mathbb{Q}$ into another curve Y over $\mathbb{Q}$ if there exists an open dense $U \subseteq X$ with an immersion $U \hookrightarrow Y$.

Non-degenerate curves

Let X be a curve over $\mathbb{Q}$.

Theorem B gives that (each irred. component of) X has many ico models X_f.

Non-degenerate criterion:

- X is non-degenerate if X has at least one non-degenerate ico model X_f

Non-degenerate equations. If $X = V(f_1, \ldots, f_m) \subseteq \mathbb{P}^n$ is a non-degenerate curve over $\mathbb{Q}$ where $n, m \in \mathbb{N}$, then the equations

$$(f_1 = 0, \ldots, f_m = 0)$$

are an example of equations which we call non-degenerate.

As almost all X_f are non-degenerate, there exist many non-degenerate curves X over $\mathbb{Q}$. What is relation of non-degeneracy to property that X has genus[3] ≥ 2?

Theorem (C)

If X is non-degenerate then X has genus ≥ 2.

[3] For each $g \in \mathbb{N}$, the curve X has genus $\geq g$ if all irreducible components of $X_{\bar{\mathbb{Q}}}$ have geometric genus $\geq g$.

Let X be a curve over $\mathbb{Q}$. Recall the following:

Theorem (C)

If X is non-degenerate then X has genus ≥ 2.

The converse is the following open problem.

Open problem: Determine which X over $\mathbb{Q}$ of genus ≥ 2 are non-degenerate.

Known for large classes of (explicit) curves X over $\mathbb{Q}$:

- If $X = X_f$ has $\Delta \neq 0$ then X has genus ≥ 2 and X is non-degenerate.
- If $X = V(F) \subset \mathbb{A}^2$ satisfies the explicit geometric criterion (τ) then X has genus ≥ 2 and X is non-degenerate. We shall explain criterion (τ) later.

Let X be an irreducible (and reduced) curve over $\mathbb{Q}$. Assume $X \subseteq \mathbb{P}^m$ is quasi-projective over $\mathbb{Q}$ and let h be the usual logarithmic Weil height on $\mathbb{P}^m$.

Theorem (D)

Suppose X is non-degenerate. For any non-degenerate ico model X_f of X, there exists a controlled open dense $U \subseteq X$ such that all $x \in U(\mathbb{Q})$ satisfy

$$h(x) \leq c \cdot d_X \nu_f^{\kappa} + h_X, \quad \nu_f = \mathrm{rad}(\Delta), \quad c = 10^{10^{12}}, \quad \kappa = 24.$$

$d_X = \deg(\tilde{X} \to X)\deg(\overline{X})$ for $\tilde{X} \to X$ normalization and $\overline{X} \subseteq \mathbb{P}^m$ closure

$h_X =$ height of $\tilde{X}$ inside $\mathbb{P}^4 \times \mathbb{P}^m$

$U \subseteq X$ is largest open of X with an immersion $U \hookrightarrow X_f$

Application. To deduce from Theorem (D) explicit Weil height bounds for all $x \in X(\mathbb{Q})$, one can proceed as follows:

(i) Construct a non-degenerate ico model X_f of X with ν_f controlled.

(ii) Control $h(x)$ for the finitely many $x \in (X \setminus U)(\bar{\mathbb{Q}})$.

We emphasize that (i) and (ii) are both geometric problems. To illustrate the application of Theorem (D) via (i) and (ii) we will consider a special class of non-degenerate plane curves satisfying a simple non-degeneracy criterion (τ).

6. Equations $F(x, y) = 0$

Criterion (τ) for plane curves

Motivation. We introduce a simple non-degeneracy criterion for plane curves which allows to explicitly produce large classes of non-degenerate plane curves.

Let $F \in \mathbb{Z}[x, y]$ be of degree $d \geq 1$ and consider $X = V(F) \subset \mathbb{A}^2 \subset \mathbb{P}^2$.

Criterion (τ)**.** Let $\tau : \mathbb{P}^2 \dashrightarrow \mathbb{P}^4$ be the explicit rational map over $\mathbb{Q}$ defined in the next slide, and let $e_i \in \mathbb{P}^4$ be the five permutations of $e_1 = (1, 0, \ldots, 0)$.

Criterion (τ) The closed image $\overline{\tau(X)}$ contains no e_i.

The rational map $\tau : \mathbb{P}^2 \dashrightarrow \mathbb{P}^4$ over $\mathbb{Q}$ is birational onto its image and for any given F one can compute (over $\mathbb{Q}$ or $\mathbb{C}$) whether (τ) holds.

Theorem (E)

If $X = V(F)$ satisfies (τ) then X is non-degenerate.

Non-degenerate equations. For each $F \in \mathbb{Z}[x, y]$ such that $V(F) \subset \mathbb{A}^2$ satisfies (τ), Theorem (E) implies that the equation

$$F^h = 0, \quad F^h \in \mathbb{Z}[x, y, z] \text{ homogenization},$$

is an example of an equation which we call non-degenerate.

Criterion (τ) for plane curves: Definition of τ

We define homogeneous polynomials τ_i in $\mathbb{Z}[x, y, z]$ of degree 12 as follows: We put $\tau_i = -(\prod_{j \neq i} t_j)(\sum t_j)$ for $i \in \{0, 1, 2, 3\}$ and $\tau_4 = \prod t_j$, where

$$t_0 = (y - z)(xy + xz - z^2), \qquad t_1 = xz^2 + yz^2 - x^2y - z^3,$$
$$t_2 = x(z^2 - y^2 - xz), \qquad t_3 = z(yz - xz + x^2 - y^2).$$

The five τ_i define a morphism $\tau : \mathbb{P}^2 \setminus T_\tau \to \mathbb{P}^4$ where $T_\tau = \cap V_+(\tau_i)$ is finite.

Explicit classes of plane curves satisfying criterion (τ).

One can explicitly construct large classes of plane curves satisfying criterion (τ). To illustrate this, we take $n \in \mathbb{Z}_{\geq 1}$ and we define

$$r = \begin{cases} 5 & \text{if } n = 1, \\ 4n^2 - 4n + 6 & \text{if } n \geq 2. \end{cases}$$

Theorem (F)

For each $n \in \mathbb{Z}_{\geq 1}$ there exist r explicit polynomials $F_i \in \mathbb{Q}[x, y]$ such that:

- *The r polynomials $F_i \in \mathbb{Q}[x, y]$ are $\mathbb{Q}$-linearly independent.*
- *For all $a \in \mathbb{Q}^r$ with $\prod_{i=1}^{5} a_i \neq 0$, the curve $V(\sum a_i F_i) \subset \mathbb{A}^2$ satisfies (τ).*

The F_i can be computed directly as follows: One can simply take

$$F_i = \tau^* f_i(x, y, 1), \quad \tau^* f_i = f_i(\tau_0, \ldots, \tau_4) \in \mathbb{Q}[x, y, z],$$

where $f_i \in \mathbb{Q}[x_0, \ldots, x_4]$ are monomials of degree n forming a $\mathbb{Q}$-basis of the n-th graded part of $\mathbb{Q}[x_0, \ldots, x_4]/(\sigma_2, \sigma_4)$ such that $f_i = x_{i-1}^n$ for $i \leq 5$.

Height bounds for rational points on plane curves: Conjecture (EM)

Let $F \in \mathbb{Z}[x, y]$ be of degree $d \geq 1$ with coefficients a_ι, and set $|F| = \max |a_\iota|$. If $X = V(F) \subset \mathbb{A}^2$ has genus ≥ 2, then $X(\mathbb{Q}) = \{(x, y) \in \mathbb{Q}^2;\ f(x, y) = 0\}$ is finite by the Mordell conjecture proven by Faltings (1983). This motivates:

Conjecture (EM). *If $X = V(F) \subset \mathbb{A}^2$ has genus ≥ 2 then each $(x, y) \in \mathbb{Q}^2$ with $F(x, y) = 0$ satisfies $\max(h(x), h(y)) \leq c_d |F|^{\kappa_d}$ for effective c_d, κ_d.*

Here effective means $c_d = \exp^{(\circ n)}(d)$ for an explicit n, while h is usual logarithmic Weil height: $h(x) = \log \max(|m|, |n|)$ for $x = \frac{m}{n} \in \mathbb{Q}$, $m, n \in \mathbb{Z}$ coprime.

Height bounds for points on plane curves: Some known results

Integral analogue.

- After Baker's breakthrough (1966), many authors used logarithmic forms to establish the S-integral analogue (e.g. $(x, y) \in \mathbb{Z}_S^2$) of Conjecture (EM) for large classes of (explicit) curves over any number field K; see e.g. the excellent books of Baker–Wüstholz[4] and Evertse–Győry[5][6] for overviews.
- Recently Corvaja–Lombardo–Zannier (2024) made important progress on this S-integral analogue for large classes of genus two curves over any K.

Under certain rank assumptions.

- In case certain rank assumptions are satisfied, there are powerful methods (e.g. Chabauty's method,...) giving very strong Diophantine results for $X(\mathbb{Q}) = \{(x, y) \in \mathbb{Q}^2,\ F(x, y) = 0\}$ which often allow to determine $X(\mathbb{Q})$.
- These methods usually work very well in practice for a given F.
- Some of these methods also work for certain families of F. For example, in a series of papers, Viada, Checcoli–Veneziano–Viada and Veneziano–Viada established over the last years a sharper version of Conjecture (EM) for large families of (explicit) curves over any K of growing genus. Their proofs build on the theory of anomalous intersections of Bombieri, Masser, Zannier, and their bounds allow them to determine $X(\mathbb{Q})$.

[4] Logarithmic Forms and Diophantine Geometry
[5] Unit Equations in Diophantine Number theory
[6] Discriminant equations in Diophantine Number theory

Proof of Conjecture (EM) for X satisfying (τ) via Theorems (D) and (E)

Suppose $X = V(F)$ satisfies (τ). Then X is non-degenerate by Theorem (E).

Theorem (D): *For any non-degenerate ico model X_f of X, there exists a controlled open dense $U \subseteq X$ such that all $x \in U(\mathbb{Q})$ satisfy*

$$h(x) \leq c \cdot d_X \nu_f^{\kappa} + h_X, \quad \nu_f = \mathrm{rad}(\Delta), \quad c = 10^{10^{12}}, \quad \kappa = 24.$$

Application. To deduce explicit height bounds for all $x \in X(\mathbb{Q})$:

(i) Construct a non-degenerate ico model X_f of X with ν_f controlled.

(ii) Control $h(x)$ for the finitely many $x \in (X \setminus U)(\bar{\mathbb{Q}})$.

For (i) combine algebraic geometry with theory of heights of projective varieties, while for (ii) exploit the explicit τ to get $h(x) \leq 10$ for all $x \in (X \setminus U)(\bar{\mathbb{Q}})$.

Corollary. *If $X = V(F)$ satisfies (τ), then the curve X over $\mathbb{Q}$ is non-degenerate and Conjecture (EM) holds with $\kappa_d = 8^8 d^2$ and $c_d = 8^{d\kappa_d^2}$.*

Theorem (F) constructs large classes of explicit plane curves satisfying (τ): For each $n \in \mathbb{Z}_{\geq 1}$ the space $\mathbb{A}^r \setminus \left(\cup_{i=1}^5 V(z_i)\right)$ of dimension $r \sim 4n^2$ parametrizes such curves of degree $12n$, and for each odd square $g \geq 2$ there exists a moduli space of dimension $\sim g$ which parametrizes such curves of geometric genus g.

Thank you very much for your attention!