On the unique solvability of simultaneous Pell equations.

Volker Ziegler

University of Salzburg

23rd of May 2025
Online Number Theory Seminar

Pell equations I

Let $d > 1$ be a given positive integer that is not a perfect square. Then it is well known that all solutions $(X, Y) \in \mathbb{Z}^2$ of the Diophantine equation

$$X^2 - dY^2 = 1$$

can be obtained from the formula

$$X + Y\sqrt{d} = \pm\epsilon^k, \qquad k \in \mathbb{Z}$$

where $\epsilon = X_0 + Y_0\sqrt{d}$ and $(X_0, Y_0) \in \mathbb{Z}^2$ is the solution with $X_0, Y_0 > 0$ and Y_0 is minimal among all such solutions. We call $\epsilon = X_0 + Y_0\sqrt{d}$ the fundamental solution.

Pell equations II

It is also well known that such a fundamental solution ϵ always exists.

Pell equations II

It is also well known that such a fundamental solution ϵ always exists.
However, for d large (say $d > 10^7$) it is quite a computional challenge to find a fundamental solution for given d, unless d is of some special form like $d = n^2 \pm 1$ for some integer n.

Pell equations II

It is also well known that such a fundamental solution ϵ always exists.
However, for d large (say $d > 10^7$) it is quite a computional challenge to find a fundamental solution for given d, unless d is of some special form like $d = n^2 \pm 1$ for some integer n.
If we have found a fundamental solution ϵ to $X^2 - dY^2 = 1$, then we have

$$X = \frac{\epsilon^k + \epsilon^{-k}}{2} \qquad Y = \frac{\epsilon^k - \epsilon^{-k}}{2\sqrt{d}}.$$

Systems of Pell equations

Let $a > b \geq 2$ be integers. In this talk we consider the system

$$X^2 - a\,Y^2 = 1 \qquad Z^2 - b\,X^2 = 1$$

of Pell equations.

Systems of Pell equations

Let $a > b \geq 2$ be integers. In this talk we consider the system

$$X^2 - a\,Y^2 = 1 \qquad Z^2 - b\,X^2 = 1$$

of Pell equations.
Assume that neither a nor b are both perfect squares. Then each individual equation has infinitely many solutions.
However, the system has at most finitely many solutions.
The question remains how many solutions?

Systems of Pell equations

Let $a > b \geq 2$ be integers. In this talk we consider the system

$$X^2 - a\,Y^2 = 1 \qquad Z^2 - b\,X^2 = 1$$

of Pell equations.

Assume that neither a nor b are both perfect squares. Then each individual equation has infinitely many solutions. However, the system has at most finitely many solutions. The question remains how many solutions?

Conjecture

The system of Pell equations

$$X^2 - a\,Y^2 = 1 \qquad Z^2 - b\,X^2 = 1$$

has at most one solution in positive integers.

What do we know

The system of Pell equations

$$X^2 - aY^2 = 1 \qquad Z^2 - bX^2 = 1,$$

with $X, Y, Z > 0$ has

- at most two solutions (Cipu, Mignotte 2007);

What do we know

The system of Pell equations

$$X^2 - aY^2 = 1 \qquad Z^2 - bX^2 = 1,$$

with $X, Y, Z > 0$ has

- at most two solutions (Cipu, Mignotte 2007);
- has at most one solution if $b = m^2 - 1$ for some integer m (Yuan 2002, Cipu 2007);

What do we know

The system of Pell equations

$$X^2 - aY^2 = 1 \qquad Z^2 - bX^2 = 1,$$

with $X, Y, Z > 0$ has

- at most two solutions (Cipu, Mignotte 2007);
- has at most one solution if $b = m^2 - 1$ for some integer m (Yuan 2002, Cipu 2007);
- All solutions are knonw if $b = 24$ and a is a prime (Ai, Chen, Zhang, Hu 2015);

What do we know

The system of Pell equations

$$X^2 - a\,Y^2 = 1 \qquad Z^2 - b\,X^2 = 1,$$

with $X, Y, Z > 0$ has

- at most two solutions (Cipu, Mignotte 2007);
- has at most one solution if $b = m^2 - 1$ for some integer m (Yuan 2002, Cipu 2007);
- All solutions are knonw if $b = 24$ and a is a prime (Ai, Chen, Zhang, Hu 2015);
- All solutions are known if $b = m^2 - 1$ and $a = pq$ or $a = 2pq$ (Cipu 2018, Jiang 2020).

Main result

Most of the results require the special form that b is almost a square, e.g. that $b = m^2 \pm 1$.

Theorem (Hilgart, Z. 2024)

Let $a > b \geq 2$ be two positive integers, where b is fixed. Then there exists an effectively computable constant a_0, dependent only on b, such that for $a \geq a_0$ the simultaneous Pell equations

$$X^2 - a\,Y^2 = 1, \qquad Z^2 - b\,X^2 = 1 \tag{1}$$

have at most one solution (X, Y, Z) in positive integers.

Note that the bound a_0 is typically very large. However, one can use reduction methods to resolve the conjecture for any given (not too large) b:

Note that the bound a_0 is typically very large. However, one can use reduction methods to resolve the conjecture for any given (not too large) b:

Theorem (Hilgart, Z. 2024)

For $1 \leq b \leq 10\,000$, the simultaneous Pell equations

$$X^2 - a\,Y^2 = 1, \qquad Z^2 - b\,X^2 = 1,$$

have at most one solution (X, Y, Z) in positive integers for any $a \geq b$.

How to solve a system of Pell equations I

Let us assume that a and b are fixed. That is we know the fundamental solution $\epsilon = X_0 + Y_0\sqrt{a}$ of the Pell equation

$$X^2 - aY^2 = 1$$

How to solve a system of Pell equations I

Let us assume that a and b are fixed. That is we know the fundamental solution $\epsilon = X_0 + Y_0\sqrt{a}$ of the Pell equation

$$X^2 - a\,Y^2 = 1$$

And we know the fundamental solution $\delta = Z_0 + X_0\sqrt{b}$ of the Pell equation

$$Z^2 - b\,X^2 = 1.$$

How to solve a system of Pell equations I

Let us assume that a and b are fixed. That is we know the fundamental solution $\epsilon = X_0 + Y_0\sqrt{a}$ of the Pell equation

$$X^2 - a\,Y^2 = 1$$

And we know the fundamental solution $\delta = Z_0 + X_0\sqrt{b}$ of the Pell equation

$$Z^2 - b\,X^2 = 1.$$

That is any solution (X, Y, Z) of the system of Pell equations satisfies

$$\frac{\epsilon^k + \epsilon^{-k}}{2} = X = \frac{\delta^\ell - \delta^{-\ell}}{2\sqrt{b}}.$$

How to solve a system of Pell equations I

Let us assume that a and b are fixed. That is we know the fundamental solution $\epsilon = X_0 + Y_0\sqrt{a}$ of the Pell equation

$$X^2 - a\,Y^2 = 1$$

And we know the fundamental solution $\delta = Z_0 + X_0\sqrt{b}$ of the Pell equation

$$Z^2 - b\,X^2 = 1.$$

That is any solution (X, Y, Z) of the system of Pell equations satisfies

$$\frac{\epsilon^k + \epsilon^{-k}}{2} = X = \frac{\delta^\ell - \delta^{-\ell}}{2\sqrt{b}}.$$

Note that $\epsilon^k \simeq \delta^\ell b^{-1/2}$.

How to solve a system of Pell equations II

We obtain

$$\left|\epsilon^k \delta^{-\ell} \sqrt{b} - 1\right| \ll \epsilon^{-2k} \sim b\delta^{-2\ell}.$$

How to solve a system of Pell equations II

We obtain

$$\left|\epsilon^k \delta^{-\ell}\sqrt{b} - 1\right| \ll \epsilon^{-2k} \sim b\delta^{-2\ell}.$$

Since $\log(1+x) \sim x$ for x small, the above inequality implies with $x = \epsilon^k \delta^{-\ell}\sqrt{b} - 1$ the upper bound

$$\left|k \log \epsilon - \ell \log \delta + \log \sqrt{b}\right| \ll \epsilon^{-2k}.$$

How to solve a system of Pell equations II

We obtain

$$\left|\epsilon^k \delta^{-\ell}\sqrt{b} - 1\right| \ll \epsilon^{-2k} \sim b\delta^{-2\ell}.$$

Since $\log(1+x) \sim x$ for x small, the above inequality implies with $x = \epsilon^k \delta^{-\ell}\sqrt{b} - 1$ the upper bound

$$\left|k \log \epsilon - \ell \log \delta + \log \sqrt{b}\right| \ll \epsilon^{-2k}.$$

By Baker's theory of linear forms in logarithms (we apply a theorem of Matveev) we obtain

$$\exp\left(-C \log \epsilon \log \delta \log b \log k\right) \ll \left|\epsilon^k \delta^{-\ell}\sqrt{b} - 1\right| \ll \epsilon^{-2k}$$

for some (large) constant C.

How to solve a system of Pell equations III

Hence we have

$$C \log \epsilon \log \delta \log b \log k \gg k \log \epsilon$$

and we obtain

$$k \ll \log \delta \log b \log (\log \delta \log b) .$$

How to solve a system of Pell equations III

Hence we have

$$C \log \epsilon \log \delta \log b \log k \gg k \log \epsilon$$

and we obtain

$$k \ll \log \delta \log b \log \left(\log \delta \log b\right).$$

If b is fixed we get an upper bound for k. Since also a and therefore also $\log \epsilon$ is fixed we can "find" all solutions to the given system of Pell equations

$$X^2 - a\,Y^2 = 1, \qquad Z^2 - b\,X^2 = 1$$

and eventually prove that this system has at most one solution.

Two problems

If only b is fixed but a is arbitrary we run into two problems:

1. We still obtain an absolute bound for k, but we do not find a bound for ℓ, since $k \log \epsilon \simeq \ell \log \delta$.

Two problems

If only b is fixed but a is arbitrary we run into two problems:

1. We still obtain an absolute bound for k, but we do not find a bound for ℓ, since $k \log \epsilon \simeq \ell \log \delta$.
2. We still can prove that the system has at most a fixed number of solutions (the bound for k) for arbitrary a, but we are far off proving that there exists at most one solution.

Two problems

If only b is fixed but a is arbitrary we run into two problems:

1. We still obtain an absolute bound for k, but we do not find a bound for ℓ, since $k \log \epsilon \simeq \ell \log \delta$.
2. We still can prove that the system has at most a fixed number of solutions (the bound for k) for arbitrary a, but we are far off proving that there exists at most one solution.

However, we made no use that two (or more) solutions might exist.

Assumption of two solutions

Let us assume that two solutions (X_1, Y_1, Z_1) and (X_2, Y_2, Z_2) exist (without loss of generality $X_1 < X_2$) and we have two intersections:

$$\frac{\epsilon^{k_1} + \epsilon^{-k_1}}{2} = X_1 = \frac{\delta^{\ell_1} - \delta^{-\ell_1}}{2\sqrt{b}}$$

and

$$\frac{\epsilon^{k_2} + \epsilon^{-k_2}}{2} = X_2 = \frac{\delta^{\ell_2} - \delta^{-\ell_2}}{2\sqrt{b}}.$$

Two linear forms in logarithms

We obtain now two linear forms in logarithms:

$$\left|k_1 \log \epsilon - \ell_1 \log \delta + \log \sqrt{b}\right| \ll b\delta^{-2\ell_1}$$

and

$$\left|k_2 \log \epsilon - \ell_2 \log \delta + \log \sqrt{b}\right| \ll b\delta^{-2\ell_2}.$$

Two linear forms in logarithms

We obtain now two linear forms in logarithms:

$$\left|k_1 \log \epsilon - \ell_1 \log \delta + \log \sqrt{b}\right| \ll b\delta^{-2\ell_1}$$

and

$$\left|k_2 \log \epsilon - \ell_2 \log \delta + \log \sqrt{b}\right| \ll b\delta^{-2\ell_2}.$$

We eliminate $\log \epsilon$ form this system of inequalities and obtain

$$\left|(k_2 - k_1) \log \sqrt{b} - (\ell_1 k_2 - \ell_2 k_1) \log \delta\right| \ll k_2\delta^{-\ell_1}.$$

Two linear forms in logarithms

We obtain now two linear forms in logarithms:

$$\left|k_1 \log \epsilon - \ell_1 \log \delta + \log \sqrt{b}\right| \ll b\delta^{-2\ell_1}$$

and

$$\left|k_2 \log \epsilon - \ell_2 \log \delta + \log \sqrt{b}\right| \ll b\delta^{-2\ell_2}.$$

We eliminate $\log \epsilon$ form this system of inequalities and obtain

$$\left|(k_2 - k_1) \log \sqrt{b} - (\ell_1 k_2 - \ell_2 k_1) \log \delta\right| \ll k_2 \delta^{-\ell_1}.$$

Note that

$$\log \delta \log b \log(\log \delta \log b) \gg k_2 > k_2 - k_1 > (\ell_1 k_2 - \ell_2 k_1).$$

Upper bounds

That is, we obtain from Baker's theory for linear forms in logarithms

$$\ell_1 \log \delta \ll \log b \log \delta \log (\log \delta \log b)$$

Upper bounds

That is, we obtain from Baker's theory for linear forms in logarithms

$$\ell_1 \log \delta \ll \log b \log \delta \log (\log \delta \log b) \ll 1.$$

Upper bounds

That is, we obtain from Baker's theory for linear forms in logarithms

$$\ell_1 \log \delta \ll \log b \log \delta \log (\log \delta \log b) \ll 1.$$

Since we have

$$\sqrt{a} < \epsilon \leq \epsilon^{k_1} \ll X_1 \ll \delta^{\ell_1} \ll 1,$$

the system of Pell equations has two solutions only if a is "small".

Final Computations – Step I

Problem

The bound a_0 for a such that for $a > a_0$ the system of Pell equations has at most one solutions is huge.

Final Computations – Step I

Problem

The bound a_0 for a such that for $a > a_0$ the system of Pell equations has at most one solutions is huge.
For $b = 24$ we get $a_0 \simeq 10^{3.3\times 10^6}$. How to solve all remaining Pell equations for a fixed b (say $b = 24$) to prove the second theorem?

Final Computations – Step I

Problem

The bound a_0 for a such that for $a > a_0$ the system of Pell equations has at most one solutions is huge.
For $b = 24$ we get $a_0 \simeq 10^{3.3\times 10^6}$. How to solve all remaining Pell equations for a fixed b (say $b = 24$) to prove the second theorem?

We assume that b is fixed.
Step 1: We use best approximation properties of continued fractions (instead of Baker's theory) and apply it to

$$\left| \frac{k_2 - k_1}{\ell_1 k_2 - \ell_2 k_1} - \frac{\log \delta}{\log \sqrt{b}} \right| \ll \delta^{-\ell_1}.$$

This yields a small bound L for ℓ_1.

Final Computations – Step I

Problem

The bound a_0 for a such that for $a > a_0$ the system of Pell equations has at most one solutions is huge.
For $b = 24$ we get $a_0 \simeq 10^{3.3\times 10^6}$. How to solve all remaining Pell equations for a fixed b (say $b = 24$) to prove the second theorem?

We assume that b is fixed.
Step 1: We use best approximation properties of continued fractions (instead of Baker's theory) and apply it to

$$\left|\frac{k_2 - k_1}{\ell_1 k_2 - \ell_2 k_1} - \frac{\log \delta}{\log \sqrt{b}}\right| \ll \delta^{-\ell_1}.$$

This yields a small bound L for ℓ_1.

Final Computations – Step II

Step 2: For each ℓ between 1 and L we compute $x_\ell = \frac{\delta^\ell - \delta^{-\ell}}{2\sqrt{b}}$. Each x_ℓ is a candidate for X, such that (X, Y, Z) is a solution to

$$X^2 - a\,Y^2 = 1, \qquad Z^2 - b\,X^2 = 1.$$

Final Computations – Step III

Step 3: We compute for each ℓ between 1 and L the quantity

$$\gamma_\ell = x_\ell + \sqrt{x_\ell^2 - 1}.$$

Final Computations – Step III

Step 3: We compute for each ℓ between 1 and L the quantity

$$\gamma_\ell = x_\ell + \sqrt{x_\ell^2 - 1}.$$

If x_ℓ is part of a solution to the system of Pell equations

$$X^2 - a\,Y^2 = 1, \qquad Z^2 - b\,X^2 = 1,$$

then we have $x_\ell^2 - 1 = ay_\ell^2$. That is

$$\gamma_\ell = x_\ell + y_\ell\sqrt{a}$$

yields a solution to $X^2 - aY^2 = 1$ and $\gamma_\ell = \epsilon^\kappa$.

Final Computations – Step III

Step 3: We compute for each ℓ between 1 and L the quantity

$$\gamma_\ell = x_\ell + \sqrt{x_\ell^2 - 1}.$$

If x_ℓ is part of a solution to the system of Pell equations

$$X^2 - aY^2 = 1, \qquad Z^2 - bX^2 = 1,$$

then we have $x_\ell^2 - 1 = ay_\ell^2$. That is

$$\gamma_\ell = x_\ell + y_\ell\sqrt{a}$$

yields a solution to $X^2 - aY^2 = 1$ and $\gamma_\ell = \epsilon^\kappa$.
If we compute the square-free part s of $x_\ell^2 - 1 = ay_\ell^2$ we obtain a lower bound for a.

Final Computations – Step III

Step 3: We compute for each ℓ between 1 and L the quantity

$$\gamma_\ell = x_\ell + \sqrt{x_\ell^2 - 1}.$$

If x_ℓ is part of a solution to the system of Pell equations

$$X^2 - a\,Y^2 = 1, \qquad Z^2 - b\,X^2 = 1,$$

then we have $x_\ell^2 - 1 = ay_\ell^2$. That is

$$\gamma_\ell = x_\ell + y_\ell\sqrt{a}$$

yields a solution to $X^2 - aY^2 = 1$ and $\gamma_\ell = \epsilon^\kappa$.
If we compute the square-free part s of $x_\ell^2 - 1 = ay_\ell^2$ we obtain a lower bound for a.
We obtain $\kappa \leq 2\frac{\log\gamma}{\log s} := M$. Note that $\epsilon > \sqrt{a} \geq \sqrt{s}$.

Final Computations – Step IV

Step 4: We replace $\log \epsilon$ by $\frac{1}{\kappa} \log \gamma_\ell$ in one of the linear forms in logarithms and obtain:

$$\left| k_2 \log \gamma_\ell - \kappa \ell_2 \log \delta + \kappa \log \sqrt{b} \right| \ll \delta^{-2\ell_2}.$$

Using the "Baker-Davenport reduction" we obtain an upper bound for ℓ_2 for every possible $\kappa \leq M$.

Final Computations – Step IV

Step 4: We replace $\log \epsilon$ by $\frac{1}{\kappa} \log \gamma_\ell$ in one of the linear forms in logarithms and obtain:

$$\left| k_2 \log \gamma_\ell - \kappa \ell_2 \log \delta + \kappa \log \sqrt{b} \right| \ll \delta^{-2\ell_2}.$$

Using the “Baker-Davenport reduction” we obtain an upper bound for ℓ_2 for every possible $\kappa \leq M$.
That is we find a small upper bound $\tilde{L}$ for ℓ_2.

Final Computations – Step V

Step 5: For all $1 \leq \ell \leq L$ and $1 \leq \ell' \leq \tilde{L}$ we check if x_ℓ and $x_{\ell'}$ are part of two solutions to the system of Pell equations

$$X^2 - a\,Y^2 = 1, \qquad Z^2 - b\,X^2 = 1.$$

Final Computations – Step V

Step 5: For all $1 \leq \ell \leq L$ and $1 \leq \ell' \leq \tilde{L}$ we check if x_ℓ and $x_{\ell'}$ are part of two solutions to the system of Pell equations

$$X^2 - a\,Y^2 = 1, \qquad Z^2 - b\,X^2 = 1.$$

If they are part of two solutions we have

$$(x_\ell^2 - 1)(x_{\ell'}^2 - 1) = ay_\ell^2 ay_{\ell'}^2 = (ay_\ell y_{\ell'})^2 .$$

Final Computations – Step V

Step 5: For all $1 \leq \ell \leq L$ and $1 \leq \ell' \leq \tilde{L}$ we check if x_ℓ and $x_{\ell'}$ are part of two solutions to the system of Pell equations

$$X^2 - a\,Y^2 = 1, \qquad Z^2 - b\,X^2 = 1.$$

If they are part of two solutions we have

$$(x_\ell^2 - 1)(x_{\ell'}^2 - 1) = ay_\ell^2 ay_{\ell'}^2 = (ay_\ell y_{\ell'})^2\,.$$

That is we can check numeically whether

$$\sqrt{(x_\ell^2 - 1)(x_{\ell'}^2 - 1)}$$

is an integer or not. Only if it is an integer, two solutions exist.

Computation time

We implemented this idea in Sage.

- Checking all (non-square) b in the range from 1 to 10 000 took approximately 100 hours on a standard desktop PC.

Computation time

We implemented this idea in Sage.

- Checking all (non-square) b in the range from 1 to 10 000 took approximately 100 hours on a standard desktop PC.
- A single b took about 37 seconds on average, while the longest run was around 196 seconds.

Thank you for your Attention!