$\xi$ alg. number of degree $d$.

$\xi = [a_0; a_1, a_2, \dots]$ $\qquad$ $p_k/q_k = [a_0; a_1, \dots, a_k]$

<u>Fact</u>: $\xi$ is quadratic $\iff$ $(a_m)_{m \geq 1}$ is ult. periodic

$$\iff \quad q_{k+2s} - t\, q_{k+s} + (-1)^s q_k = 0, \quad \forall k \geq r$$

$$\left( \xi = [a_0; a_1, \dots, a_{r-1}, \overline{b_0, \dots, b_{s-1}}] \right)$$

About the growth:

Liouville $\rightsquigarrow$ $q_{k+1} \ll q_k^{d-1}$

$$\frac{\log q_{k+1}}{\log q_k} \leq d \quad \rightsquigarrow \quad \log \log q_k = O(k)$$

<u>Roth</u>: $\varepsilon > 0$ $\quad q_{k+1} \ll q_k^{1+\varepsilon}$

$$\frac{\log q_{k+1}}{\log q_k} \leq 1+\varepsilon \quad \rightsquigarrow \quad \log\log q_k = o(k)$$

<u>Quantitative</u> Roth's th.

$$\frac{\log q_{k+1}}{\log q_k} \leq 1+\varepsilon \quad \text{except for } \varepsilon^{-4} \text{ values of } k$$

$$\hookrightarrow \quad \log q_k \leq (1+\varepsilon)^k d^{\varepsilon^{-4}}$$

$$\log\log q_k \leq \cancel{k \log} \, k\,\varepsilon + \varepsilon^{-4} \log d$$

$$\varepsilon \simeq k^{-1/5} \qquad \log\log q_k \ll k^{4/5}$$

Consequence of Ridout's th:

$P[p_k]$, $P[q_k] \xrightarrow[k\to+\infty]{} +\infty$

no rate of growth ... $P[\,.\,]$: greatest prime divisor.

$S$ : finite set of primes

$$[n]_S = \prod_{\ell \in S} \ell^{v_\ell(n)}$$

$[p_k]_S < p_k^{\varepsilon}$

$[q_k]_S < q_k^{\varepsilon}$ if $k \geq k_0(\varepsilon)$

Effective result (Shorey, 1983):

$$P[p_k q_k] \gg_{\text{eff.}} \log\log q_k \cdot \frac{\log\log\log q_k}{\log\log\log\log q_k}$$

$P[q\langle q\xi\rangle]$

$\lfloor \cdot \rfloor$ : nearest integer function

$q \lfloor q\xi \rfloor$ $\qquad p := \lfloor q\xi \rfloor$

$$0 < |\xi - p/q| \leq 1/q$$

$p/q = l_1^{u_1} \cdots l_t^{u_t}$ $\qquad l_1, \dots, l_t$ : the first $t$ primes.

$p/q\, \xi^{-1} - 1$ is very small

$$\log |p/q\, \xi^{-1} - 1| \gg -C^t \prod_{j=1}^{t} (\log l_j) \log u$$

$\Uparrow$ $\qquad u \ll \log q$

$-\log q$

$$\log q \ll C^t \prod_{j=1}^{t} (\log l_j) \log \log q$$

We can do better when $d = 2$

$$P[q_k] \gg (\log q_k)^{1/2} \exp\left(\frac{\log\log q_k}{105 \log\log\log q_k}\right).$$ (Stewart)

$[p_k q_k]_S \leq (p_k q_k)^{1-\tau}$ for some $\tau > 0$ : effective result.

$\cancel{[p_k q_k]_S}$ $[p_k]_S$ , $[q_k]_S \leq q_k^{\varepsilon}$ , $\forall k \geq k_0(\varepsilon)$.

<u>Mahler</u> : $\varepsilon > 0$ $\qquad$ $\| \cdot \|$ : distance to the nearest integer.

$$\left\| \left(\tfrac{3}{2}\right)^n \right\| > \frac{1}{2^{\varepsilon n}} \quad , \ \forall\, n > n_0(\varepsilon)$$

<u>ineff.</u>

Actually, all the partial quotients are $< 2^{\varepsilon n}$.

Ridout's th.

(Pourchet, van der Poorten)

Consequence :

$$\mathcal{L}\left(\left(\tfrac{3}{2}\right)^n\right) \xrightarrow[n \to +\infty]{} +\infty$$

$\mathcal{L}(\cdot)$ : length of the continued fr. expansion.

Stronger tool: Schmidt Subspace th

Corvaja - Zannier 2004, 2005.

(i) $\mathcal{L}\left(\frac{3^n+1}{2^n+1}\right), \mathcal{L}\left(\frac{17^n+3^n+1}{11^n+5^n+2}\right) \xrightarrow[n\to+\infty]{} +\infty$

(ii) $\alpha$ quadratic real nb, $\alpha \neq \sqrt{a/b}$, $\alpha \neq$ unit in $\mathbb{Q}(\alpha)$

Then, the length of the period of the c.f. expansion of $\alpha^n$ tends to infinity with $n$.

In the same paper :

$\alpha > 1$ , alg. real nb such that

$\alpha^d$ is <u>not</u> a Pisot nb for any $d \geq 1$.

$\forall \varepsilon > 0$. Then $\| \alpha^n \| > 2^{-\varepsilon n}$ , $\forall n \geq n_0(\varepsilon)$.

$\gamma$ : Golden Ratio $\| \gamma^n \| \asymp \gamma^{-n}$

This result has been extended by

Kulkarni, Mavraki, Nguyen to

$\| b_1 \alpha_1^n + \dots + b_h \alpha_h^n \| > \dots$ (technical assumptions).

$\deg \xi = 2$ : $(q_k)$ is a union of binary recurrences.

$(u_n)_{n\geq 1}$ : linear recurrence.

Is the intersection $(q_k)_{k\geq 1} \cap (u_n)_{n\geq 1}$ finite?
(when $d \geq 3$)

$u_n = n$

If $(u_n)_n$ is non-degenerate, then the answer is YES! (B. - Nguyen, 2022).

Open question: $(q_k)_{k\geq 1} \cap (n^2)_{n\geq 1}$ finite?

We proved that $\varepsilon > 0$

$$\left\{ n : u_n \neq 0 \ \& \ \| u_n \xi \| < \frac{1}{|u_n|^{\frac{1}{d-1} + \varepsilon}} \right\}$$

is finite.

For $d \geq 3$, $\frac{1}{d-1} + \varepsilon \leq \frac{3}{4}$ $\qquad \| q_k \xi \| < \frac{1}{q_k}$

$\varepsilon = 1/4$

For $d = 3$, the exponent $\frac{1}{d-1} = \frac{1}{2}$ is best possible.

Rem.: The case where $(u_n)_n$ has integral roots $> 1$ and rational coeff$\underline{s}$ was proved by Corvaja & Zannier in their book.

Lenstra & Shallit



<u>Cor.</u>: $\xi$ irrat. real nb, $(p_k)$, $(q_k)$
The 4 conditions are equivalent:

(i) $(p_k)_{k\geq 1}$ has an infinite intersection with some non-degenerate linear recurrence that is not a polynomial seq.

(ii) $(q_k)_{k\geq 1}$ ...

(iii) $(a_\ell)_{\ell\geq 1}$ is ult. periodic

(iv) $\xi = [a_0, a_1, \dots]$ is quadratic irrat.

Question: Does the number of nonzero digits in the base-$b$ representation of $q_k(\xi)$ tends to infinity with $n$? (12)

(B. Nguyen) Yes! Nearly immediate conseq. of the Schmidt Subspace th.

Pb: is it possible to quantify this result?

Stewart 1980, case $d=2$:

$$\mathcal{N}(q_k, b) > \frac{\lg k}{\lg\lg k}$$

$\mathcal{N}(N, b)$ is the nb of nonzero digits in the base-$b$ repres. of $N$.

$P[q_k] \xrightarrow[k \to +\infty]{} +\infty$, effective proof?

$$\xi \quad , \quad \zeta$$
$$\downarrow \qquad \downarrow$$
$$(q_k) \qquad (\tilde{q}_k)$$

$\xi$, $\zeta$ algebraic

$$\begin{cases} |q_k \xi - p_k| < \dfrac{1}{q_k} \\ |\tilde{q}_k \zeta - \tilde{p}_k| < \dfrac{1}{\tilde{q}_k} \end{cases}$$

$q_k = \tilde{q}_k = Q$

$$\begin{cases} |Q \xi - p_k| < \dfrac{1}{Q} \\ |Q \zeta - \tilde{p}_k| < \dfrac{1}{Q} \end{cases}$$

$1, \xi, \zeta$ are linearly dependent over $\mathbb{Q}$.
(by the Subspace Theorem)