Bounds for the number of distinct squares in binary recurrence sequences

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Introduction

- Recurrence sequences are ubiquitous in maths and nature [12].
  Arithmetic properties of important sequences are valuable.
  Eg. 0, 1, 8, 144: all powers in Fibonacci sequence (BMS 2006 [2]).
- Squares related to $aX^2 - bY^4 = c$ (quartic model of elliptic curve)
  Long history: Mordell, Ljunggren,...
  Sharp bounds for number of integer solutions for $c = \pm 1, \pm 2, \pm 4$.
  See Akhtari 2009 [1] and references there.
- Other values of c? Not a lot known.
- $X^2 - \left(2^{2m} + 1\right) Y^4 = -2^{2m}$: ≤ 12 odd positive integer solutions.
  He, Togbé and Walsh 2009 [4].
- Improved to at most 3 such solutions.
  Stoll, Walsh and Yuan 2009 [5].
- Uses hypergeometric method.

New Results (I): $X^2 - \left(a^2+b\right) Y^4 = -b$

Theorem (V. 2024, Corollary 1.3, ref. 8)

Let a, m and p be non-negative integers with $a \geq 1$, p a prime and put $b = p^m$ or $4p^m$. Suppose that $\gcd\left(a^2, b\right)$ is squarefree, a^2+b is not a square and that $x^2 - \left(a^2+b\right) y^2 = -4$ has a solution in positive integers.

(a) *If b is a square, then $X^2 - \left(a^2+b\right) Y^4 = -b$ has at most two coprime positive integer solutions.*

(b) *If b is not a square, then $X^2 - \left(a^2+b\right) Y^4 = -b$ has at most three coprime positive integer solutions.*

- Generalises and improves SWY result ($a = 1$ and $b = 2^{2m}$).
- Also best possible:
  Let $b > 5$, with $b \equiv 1 \pmod 4$ and $5 \nmid b$, $a = (b-5)/4$.
  $(X, Y) = (a, 1)$, $\left(\left(b^3 + 5b^2 + 15b - 5\right)/16, (b+1)/2\right)$.
- Proof is actually about squares in binary recurrence sequences.

Our Sequences

- a, b and d positive integers, d not a square.
  $\alpha = a + b^2\sqrt{d}$ with norm $N_\alpha = a^2 - b^4 d$,
  $\varepsilon = \left(t + u\sqrt{d}\right)/2$ a unit in $\mathcal{O}_{\mathbb{Q}(\sqrt{d})}$, t, u positive integers.
- Define $(x_k)_{k=-\infty}^{\infty}$ and $(y_k)_{k=-\infty}^{\infty}$ by

$$x_k + y_k\sqrt{d} = \alpha \cdot \varepsilon^{2k}.$$

- E.g., $\varepsilon = \left(1+\sqrt{5}\right)/2$, $\alpha = 2\varepsilon$, $x_k = L_{2k+1}$ and $y_k = F_{2k+1}$.
- Observe that $x_k^2 - dy_k^2 = N_\alpha$.
- Choose α such that $b^2 = y_0$ is the smallest square among the y_k's.

Conjecture

There are at most four distinct integer squares among the y_k's.

If $\mathrm{sf}\left(|N_\alpha|\right) | (2p)$, *where p is an odd prime, there are at most three distinct integer squares among the y_k's.*
Furthermore, if $|N_\alpha|$ is a perfect square, then there are at most two distinct integer squares among the y_k's.

- The arithmetic of N_α matters.

New Results (II): $b = 1$

Theorem (V. 2024, Theorem 1.2, ref. 8)

Let a, m and p be non-negative integers with $a \geq 1$, p a prime. Put $b = 1$ and $N_\alpha = -p^m, -2p^m, -4p^m, -8p^m$ or $-16p^m$. Suppose that $d = a^2 - N_\alpha > 0$ is not a square.
(a) *If $-N_\alpha$ is a square, then there are at most 2 distinct integer squares among the y_k's*
(b) *If $-N_\alpha$ is not a square, then there are at most 3 distinct integer squares among the y_k's.*

Theorem (V. 2024, Theorem 1.4, ref. 7)

Let $b = 1$, a and d be positive integers, where d is not a square, $N_\alpha < 0$ and $-N_\alpha$ is a square.
(a) *If $u = 1, 2$, $t^2 - du^2 = -4$ and one of $y_{\pm 1}$ is a perfect square, then there are at most 3 distinct squares among the y_k's.*
(b) *Otherwise, there are at most 2 distinct squares among the y_k's.*

New Results (III): $b \geq 1$

Let K be the largest negative integer such that $y_K > b^2 = y_0$.

Theorem (V. 2025, ref. 9, 10, 11)

(a) *Let* $-N_\alpha$ *be a positive square. There are at most two distinct squares among the* y_k*'s with* $k \geq 2$ *or* $k \leq K-1$*, and*

$$y_k > \frac{16.33 b^{8/3} |N_\alpha|^2}{\sqrt{d}}.$$

(b) *Let* $N_\alpha = -2^\ell p^m$ *with* p *an odd prime and* ℓ, m *non-negative integers. There are at most four distinct squares among the* y_k*'s with* $k \geq 2$ *or* $k \leq K-1$*, and*

$$y_k > \frac{336 b^{8/3} |N_\alpha|^2}{\sqrt{d}}.$$

(c) *Suppose* $N_\alpha < 0$*. There are at most four distinct squares among the* y_k*'s with* $k \geq 3$ *or* $k \leq K-2$*, and*

$$y_k > \frac{16 b^4 |N_\alpha|^4}{\sqrt{d}}.$$

New Results (IV): $b \geq 1$

Theorem (V. 2025, ref. 9, 10, 11)

(a) *Suppose* $-N_\alpha$ *is a positive square.*

(a-i) *For* $1 \leq b \leq 11$, *at most* 5 *distinct squares in* $(y_k)_{k=-\infty}^{\infty}$.

(a-ii) *For* $b \geq 12$, *at most* 5 *distinct squares in* $(y_k)_{k=-\infty}^{\infty}$, *if*

$$d \geq \frac{30\,|N_\alpha|^{1/2}\,b^{28/13}}{u^{24/13}}.$$

(b) $N_\alpha = -2^\ell p^m$ *with* p *an odd prime and* $\ell, m \in \mathbb{Z}_{\geq 0}$.

(b-i) *For* $1 \leq b \leq 5$, *at most* 7 *distinct squares in* $(y_k)_{k=-\infty}^{\infty}$.

(b-ii) *For* $b \geq 6$, *at most* 7 *distinct squares in* $(y_k)_{k=-\infty}^{\infty}$, *if*

$$d \geq \frac{59\,|N_\alpha|^{1/2}\,b^{28/13}}{u^{24/13}}.$$

(c) *Suppose* $N_\alpha < 0$.

(c-i) *For* $b = 1, 2, 3$, *at most* 9 *distinct squares among the* y_k*'s.*

(c-ii) *For* $b \geq 4$, *at most* 9 *distinct squares in* $(y_k)_{k=-\infty}^{\infty}$, *if*

$$d \geq \frac{15\,|N_\alpha|^{3/4}\,b^{3/2}}{u^{3/2}}.$$

Hypergeometric method crash course

- For positive integers m and n with $0 < m < n/2$ and $\gcd(m,n) = 1$ and non-negative integer r, put

$$X_{m,n,r}(z) = {}_2F_1(-r-m/n, -r, 1-m/n, z), \quad Y_{m,n,r} = z^r X_{m,n,r}\left(z^{-1}\right),$$

$$R_{m,n,r}(z) = \frac{(m/n)\cdots(r+m/n)}{(r+1)\cdots(2r+1)}\,{}_2F_1\left(r+1, r+1-m/n; 2r+2; 1-z\right),$$

where ${}_2F_1$ denotes the classical hypergeometric function.

- Key relationship:

$$z^{m/n} Y_{m,n,r}(z) - X_{m,n,r}(z) = (z-1)^{2r+1} R_{m,n,r}(z).$$

- $X_{m,n,r}(z), Y_{m,n,r}(z) \in \mathbb{Q}[z]$.
- denominators of coefficients of $X_{m,n,r}(z)$ grow like $c_1(n)c_2(n)^r$.
- $|X_{m,n,r}(z)| < c_3(n,r)\left|1+\sqrt{z}\right|^{2r}$ for $|z| \leq 1$.
- $\left|(z-1)^{2r+1} R_{m,n,r}(z)\right| \leq c_4(n,r)\left|1-\sqrt{z}\right|^{2r}$, for $|z| \leq 1$, $|z-1| < 1$.

Folklore Lemma (à la Evertse [3])

Lemma

Let $\theta \in \mathbb{C}$ and $\mathbb{K}$ an imaginary quadratic field. Suppose $k_0, \ell_0 > 0$ and $E, Q > 1$ such that for all non-negative integers r, there are algebraic integers p_r and q_r in $\mathbb{K}$ with $|q_r| < k_0 Q^r$ and $|q_r\theta - p_r| \leq \ell_0 E^{-r}$ satisfying $p_r q_{r+1} \neq p_{r+1} q_r$.

For any $p, q \in \mathcal{O}_{\mathbb{K}}$, let r_0 be the smallest positive integer such that $(Q - 1/E)\,\ell_0 |q| / (Q-1) < cE^{r_0}$, where $0 < c < 1$.

(a) *We have*

$$|q\theta - p| > \frac{1 - c/E}{k_0 Q^{r_0+1}}.$$

(b) *When $p/q \neq p_{r_0}/q_{r_0}$, we have*

$$|q\theta - p| > \frac{1 - c}{k_0 Q^{r_0}}.$$

Usually $\left|\theta - \dfrac{p}{q}\right| > \frac{1}{c|q|^{\kappa+1}}$, where $c = 2k_0 Q (2\ell_0 E)^{\kappa}$ and $\kappa = \frac{\log Q}{\log E}$.

What does this have to do with squares in sequences?

- $n \geq 3$, $D < 0$, $A, B, m \in \mathbb{Q}\left(\sqrt{D}\right)$, an integer solution, (X, Y), of
$$B\left(X + Y\sqrt{D}\right)^n - A\left(X - Y\sqrt{D}\right)^n = m$$
$\left(X + Y\sqrt{D}\right) / \left(X - Y\sqrt{D}\right)$: a good approximation to $(A/B)^{1/n}$.

- If we can associate our problem with an equation of form
$$B\left(X + Y\sqrt{D}\right)^n - A\left(X - Y\sqrt{D}\right)^n = m$$
having A/B near 1, then we can use hypergeometric method.

- Recall: $x_k + y_k\sqrt{d} = \alpha\varepsilon^{2k}$ with $y_k = y^2$ and $N_\alpha < 0$.

- We have $x + y^2\sqrt{d} = \alpha\epsilon^2$ $(\epsilon = \varepsilon^k)$,
take norm and rearrange: $x^2 - N_\alpha N_\epsilon^2 = dy^4$.

  Factoring goal: with $\widetilde{d}, \widetilde{y} \in \mathbb{Q}\left(\sqrt{N_\alpha}\right)$,
$$x + N_\epsilon\sqrt{N_\alpha} = \widetilde{d} \times \widetilde{y}^4 \text{ and } x - N_\epsilon\sqrt{N_\alpha} = \overline{\widetilde{d}} \times \overline{\widetilde{y}}^4.$$

  If so, then $\quad \widetilde{d}\widetilde{y}^4 - \overline{\widetilde{d}}\,\overline{\widetilde{y}}^4 = 2N_\epsilon\sqrt{N_\alpha}.$

Representation Lemma

Lemma (Prop 3.1, ref. 7)

Let $a, b, d \in \mathbb{Z}$ with $a \neq 0$, $b, d > 0$ and d is not a square, put $\alpha = a + b^2\sqrt{d}$. Suppose that N_α is not a square, $x \neq 0$ and $y > 0$ are rational integers with

$$x + y^2\sqrt{d} = \alpha\epsilon^2,$$

where $\epsilon = \left(t + u\sqrt{d}\right)/2 \in \mathcal{O}_{\mathbb{Q}(\sqrt{d})}$ is a unit with norm N_ϵ.

(a) *We can write*

$$\pm f^2\left(x + N_\epsilon\sqrt{N_\alpha}\right) = \left(a + \sqrt{N_\alpha}\right)\left(r + s\sqrt{\mathrm{sf}\left(N_\alpha\right)}\right)^4 \text{ and}$$

$$fy = b\left(r^2 - \mathrm{sf}\left(N_\alpha\right)s^2\right),$$

for $f, r, s \in \mathbb{Z}$, $f \neq 0$, $f \mid \left(4b^2\,\mathrm{sf}\left(|N_\alpha|\right)\right)$, $f \leq 4b^2\sqrt{\mathrm{sf}\left(|N_\alpha|\right)}$.

(b) *If $|N_\alpha|$ is a square, then $f \mid b^2$.*

(c) *If $|N_\alpha| = 2^\ell p^m$ where p is an odd prime, then we have $f \mid \left(4b^2\right)$ when $N_\alpha \equiv 1 \bmod 4$ and $f \mid \left(2b^2\right)$ otherwise.*

Proof: $\left(x + N_\varepsilon\sqrt{N_\alpha}\right)/\left(a + \sqrt{N_\alpha}\right)$ is a square in $\mathbb{Q}\left(\sqrt{N_\alpha}\right)$.
Write multiple of square root as a square, work with valuations.

Lower bound for y_k, $k \neq 0$

Lemma (Lemma 3.5, ref. 7)

Let the y_k's be defined as above. Suppose that $N_\alpha < 0$.
Let K be the largest negative integer such that $y_K > b^2 = y_0$.
(a) *For all k, $2y_k$ is a positive integer.*
(b) *The sequences $(y_k)_{k \geq 0}$ and $(y_{K+1}, y_K, y_{K-1}, y_{K-2}, \ldots)$ are increasing sequences of positive numbers.*
(c) *We have*

$$y_k \geq \begin{cases} \left(|N_\alpha|\, u^2 / \left(4b^2\right)\right) \left(2du^2/5\right)^{k-1} & \text{for } k > 0, \\ \left(|N_\alpha|\, u^2 / \left(4b^2\right)\right) \left(2du^2/5\right)^{\max(0, K-k)} & \text{for } k < 0. \end{cases}$$

Proof: induction.

- For $k \geq 3$ or $k \leq K - 2$,

$$y_k \geq \frac{|N_\alpha|\, d^2 u^6}{25 b^2}.$$

Gap Principle

Lemma (Lemma 3.3, ref. 11)

Let the y_k's be defined as before with $N_\alpha < 0$.
(a) *Suppose that $-N_\alpha$ is a square. If y_i and y_j are distinct squares with $i, j \neq 0$ and $y_j > y_i \geq \max\left\{4\sqrt{|N_\alpha|/d}, b^2|N_\alpha|/d\right\}$, then*

$$y_j > 57.32\left(\frac{d}{b^2|N_\alpha|}\right)^2 y_i^3.$$

(b) *Suppose $-N_\alpha$ is not a square. If y_{k_1}, y_{k_2} and y_{k_3} are three distinct squares with none of k_1, k_2 or k_3 between $K+1$ and 0, inclusive, and*

$$y_{k_3} > y_{k_2} > y_{k_1} \geq \max\left\{4\sqrt{|N_\alpha|/d}, \left(16b^2|N_\alpha|^2/d\right)^2/60\right\},$$

then there exist distinct $i, j \in \{k_1, k_2, k_3\}$ such that

$$y_j > 1.43\frac{d}{b^2|N_\alpha|^2} y_i^{5/2}.$$

E.g., $a = 140$, $b = 1$, $d = 48024901$, $t = 13860$, $u = 2$, $N_\alpha = -3 \cdot 23 \cdot 41 \cdot 71 \cdot 239$, $y_{-1} = 9701^2$, $y_1 = 9899^2$.

Proof Strategy

- For $b = 1$ and $-N_\alpha$ square: assume two distinct squares, $y_\ell > y_k > 1$, satisfying $k, \ell \neq 0$.
- apply gap principle once.
- For any b and any $N_\alpha < 0$: assume five distinct squares, $y_{k_5} > y_{k_4} > y_{k_3} > y_{k_2} > y_{k_1}$ satisfying $k_1 \geq 3$ or $k_1 \leq K - 2$.
- allows us to apply gap principle twice: m_1, m_2 and m_3.
- $\omega_{m_1} = \left(x_{m_1} + N_{\varepsilon^{m_1}} \sqrt{N_\alpha} \right) / \left(x_{m_1} - N_{\varepsilon^{m_1}} \sqrt{N_\alpha} \right)$ and ζ_4 satisfies
$$\left| \omega_{m_1}^{1/4} - \zeta_4 \frac{x - y\sqrt{\mathrm{sf}\left(N_\alpha\right)}}{x + y\sqrt{\mathrm{sf}\left(N_\alpha\right)}} \right| = \min_{0 \leq j \leq 3} \left| \omega_{m_1}^{1/4} - e^{2j\pi i/4} \frac{x - y\sqrt{\mathrm{sf}\left(N_\alpha\right)}}{x + y\sqrt{\mathrm{sf}\left(N_\alpha\right)}} \right|$$
where
$$q = x + y\sqrt{\mathrm{sf}\left(N_\alpha\right)} = \left(r_{m_1} - s_{m_1}\sqrt{\mathrm{sf}\left(N_\alpha\right)} \right) \left(r_{m_3} + s_{m_3}\sqrt{\mathrm{sf}\left(N_\alpha\right)} \right)$$
with r_{m_1}, s_{m_1} and r_{m_3}, s_{m_3} as in the Representation Lemma.
- Put $p = \overline{q}$ and apply "bucketing" from Folklore Lemma.

Step 1: $r_0 = 1$ and $\zeta_4 p/q \neq p_{r_0}/q_{r_0}$

Folklore Lemma with $|q| = \sqrt{f_{m_1} f_{m_3}} \left(y_{m_1} y_{m_3} \right)^{1/4} / b$ and $c = 0.75$,

$$\frac{0.99b}{k_0 Q^{r_0} \sqrt{f_{m_1} f_{m_3}} \left(y_{m_1} y_{m_3} \right)^{1/4}} < 3.96 \left| \omega_{m_1}^{1/4} - \zeta_4 \frac{x - y\sqrt{\mathrm{sf}\left(N_\alpha\right)}}{x + y\sqrt{\mathrm{sf}\left(N_\alpha\right)}} \right| < \frac{2\sqrt{|N_\alpha|}}{\sqrt{d}\, y_{m_3}}$$

Since $|q_{r_0}| < k_0 Q^{r_0}$ with $Q < 10.74\sqrt{d}\, y_{m_1}$ and $k_0 < 0.89$,

$$\left(\frac{3.24 N_\alpha f_{m_1} f_{m_3}}{b^2} \right)^2 10.74^{4r_0} d^{2r_0 - 2} y_{m_1}^{4r_0+1} > y_{m_3}^3.$$

For $r_0 = 1$, with $f_{m_1} f_{m_3} \leq 16 b^4 |N_\alpha|$,

$$y_{m_3} < 331 N_\alpha^{4/3} b^{4/3} y_{m_1}^{5/3}.$$

Applying the Gap Principle Lemma (b) twice, we have

$$y_{m_3} > \left(\frac{1.43d}{b^2 |N_\alpha|^2} \right) y_{m_2}^{5/2} > \left(\frac{1.43d}{b^2 |N_\alpha|^2} \right)^{7/2} y_{m_1}^{25/4}.$$

Not possible if

$$y_{m_1} > \frac{2.7 b^{20/11} |N_\alpha|^{20/11}}{d^{42/55}}.$$

Step 1 (cont.): $r_0 = 1$ and $\zeta_4 p/q \neq p_{r_0}/q_{r_0}$

- Suppose
$$y_{m_1} < \frac{2.7 b^{20/11} |N_\alpha|^{20/11}}{d^{42/55}}.$$
- Since $m_1 \geq 3$ or $m_1 \leq K - 2$, from our lower bound lemma,
$$y_{m_1} \geq \frac{|N_\alpha| d^2 u^6}{25 b^2}.$$
- These bounds contradict each other when
$$d > \frac{4.6 |N_\alpha|^{45/152} b^{105/76}}{u^{165/76}}.$$
- Hence $r_0 = 1$ and $\zeta_4 p/q \neq p_1/q_1$ is not possible for such d or y_{m_1} under our assumptions.

Step 2: $r_0 = 1$ and $\zeta_4 p/q = p_{r_0}/q_{r_0}$

- For any $r_0 \geq 1$, using hypergeometric functions directly,

$$\left| \omega_{m_1}^{1/4} - \zeta_4 \frac{x - y\sqrt{\mathrm{sf}\,(N_\alpha)}}{x + y\sqrt{\mathrm{sf}\,(N_\alpha)}} \right| > \frac{0.291}{4^{r_0} \cdot r_0^{1/2}} \left(\frac{|N_\alpha|}{d} \right)^{r_0+1/2} \left(\frac{1}{y_{m_1}} \right)^{2r_0+1}.$$

- This and upper bound yield $\quad 1.73 r_0^{1/2} \left(4 \frac{d}{|N_\alpha|} \right)^{r_0} y_{m_1}^{2r_0+1} > y_{m_3}$.
- For $r_0 = 1$ and applying the gap principle twice, we obtain

$$6.92 \frac{d}{|N_\alpha|} y_{m_1}^3 > y_{m_3} > \left(\frac{1.43d}{b^2 |N_\alpha|^2} \right) y_{m_2}^{5/2} > \left(\frac{1.43d}{b^2 |N_\alpha|^2} \right)^{7/2} y_{m_1}^{25/4}.$$

- Not possible if $\quad y_{m_1} > \dfrac{1.24\, |N_\alpha|^{24/13}\, b^{28/13}}{d^{10/13}}$.
- Applying lower bound lemma to y_{m_1}, this is not possible if

$$d > \frac{3.45\, |N_\alpha|^{11/36}\, b^{3/2}}{u^{13/6}}.$$

Step 3: $r_0 > 1$ and $\zeta_4 p/q \neq p_{r_0}/q_{r_0}$

- Gap principle not good enough. Use def'n of r_0 in Folklore Lemma:
$$\sqrt{f_{m_1} f_{m_3}} \left(y_{m_1} y_{m_3} \right)^{1/4} / b = |q| > 0.86 c E^{r_0 - 1} / \ell_0.$$
- With $E > 0.366 \sqrt{d}\, y_{m_1} / |N_\alpha|$ and $\ell_0 < 0.46 \sqrt{|N_\alpha|} / |x_{m_1}|$, implies
$$y_{m_3} > \left(3.7 \sqrt{\frac{|N_\alpha|}{f_{m_1} f_{m_3}}} \right)^4 \left(\frac{0.366}{|N_\alpha|} \right)^{4r_0} d^{2r_0} y_{m_1}^{4r_0 - 1}.$$
- Recall from Step 1:
$$\left(\frac{3.24 N_\alpha f_{m_1} f_{m_3}}{b^2} \right)^2 10.74^{4r_0} d^{2r_0 - 2} y_{m_1}^{4r_0 + 1} > y_{m_3}^3.$$
- Yields $\dfrac{14.8 \cdot 67.1^{1/(2r_0 - 1)} |N_\alpha|^{3/2 + 5/(2(2r_0 - 1))} b^{7/(2r_0 - 1)}}{d^{1/2 + 1/(2r_0 - 1)}} > y_{m_1}.$

  Since $r_0 \geq 2$, simplifies to $\dfrac{61 b^{7/31} |N_\alpha|^{7/3}}{d^{1/2}} > y_{m_1}.$
- Applying lower bound lemma to y_{m_1} and $r_0 \geq 2$, not possible if
$$d > \frac{15 |N_\alpha|^{8/17} b^{20/17}}{u^{36/17}}.$$

Step 4: $r_0 > 1$ and $\zeta_4 p/q = p_{r_0}/q_{r_0}$

- From Step 3 (i.e., via definition of r_0 in Folklore Lemma):

$$y_{m_3} > \left(3.7\sqrt{\frac{|N_\alpha|}{f_{m_1} f_{m_3}}}\right)^4 \left(\frac{0.366}{|N_\alpha|}\right)^{4r_0} d^{2r_0} y_{m_1}^{4r_0-1}.$$

- From Step 2:

$$1.73 r_0^{1/2} \left(4\frac{d}{|N_\alpha|}\right)^{r_0} y_{m_1}^{2r_0+1} > y_{m_3}.$$

- Yields

$$16.33\frac{b^{2/(r_0-1)} |N_\alpha|^{3/2+3/(2(r_0-1))} \cdot 422^{1/(2(r_0-1))}}{d^{1/2+1/(2(r_0-1))}} > y_{m_1}.$$

  Since $r_0 \geq 2$, simplifies to $\dfrac{238 b^2 |N_\alpha|^3}{d^{1/2}} > y_{m_1}$.

- Applying lower bound lemma to y_{m_1} and $r_0 \geq 2$, not possible if

$$d > \frac{15 b^{4/3} |N_\alpha|^{2/3}}{u^2}.$$

Consolidation

- Get contradiction from assumption of 10 distinct squares (five y_k's with $k \geq 3$ or $k \leq K-2$ and remaining y_{K-1}, y_K, y_0, y_1, y_2) and that y_{m_1} or d is sufficiently large.
- Gap principle lemma requires, $y_{m_1} \geq 4\left|N_\alpha\right|/\sqrt{d}$ and $y_{m_1} \geq (64/15)b^4\left|N_\alpha\right|^4/d^2$.
- Combined with steps (1)–(4),

$$y_{m_1} \geq y_{k_1} > \max\left\{\begin{array}{l} \dfrac{2.7b^{20/11}\left|N_\alpha\right|^{20/11}}{d^{42/55}}, \dfrac{1.3\left|N_\alpha\right|^{24/13}b^{28/13}}{d^{10/13}}, \\ \dfrac{61b^{7/3}\left|N_\alpha\right|^{7/3}}{d^{1/2}}, \dfrac{238b^2\left|N_\alpha\right|^3}{d^{1/2}} \end{array}\right\}$$

- Hence

$$y_{k_1} > \frac{16b^4\left|N_\alpha\right|^4}{\sqrt{d}}$$

- Similarly,

$$d \geq \frac{15\left|N_\alpha\right|^{3/4}b^{3/2}}{u^{3/2}}.$$

- small b: computation. $N_\alpha < 0$ implies finitely many a for each d.

Future Directions

- Treatment of small squares among the y_k's.
  This would provide unconditional results like our conjectures.
- Better treatment of $r_0 = 1$ steps in our proof.
  Try to apply gap principle just once.
  Can the simple expressions for $r_0 = 1$ hypergeometric fcns help?
  This would yield improved results.
- Generalisation to sequences generated by ε with $|N_\varepsilon| \neq 1$.
  Possible since in Representation Lemma only depend on $\mathrm{rad}\,(N_\varepsilon)$.
- What about sequences with $N_\alpha > 0$?

Closing

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