Online Number Theory Seminar

25 April 2025. – 17:00-17:50

P. M. Voutier: Bounds for the number of distinct squares in binary recurrence sequences

Recently, I developed a technique for bounding the number of distinct squares in binary recurrence sequences. In particular, I treat sequences $(y_k)_{k=-\infty}^{\infty}$ defined as follows.
Let a, b and d be positive integers such that d is not a square. Put $\alpha = a+b^2\sqrt{d}$ with $N_{\mathbb{Q}(\sqrt{d})/\mathbb{Q}}(\alpha) < 0$ and let $\varepsilon = \left(t+u\sqrt{d}\right)/2$ be a unit in $\mathcal{O}_{\mathbb{Q}(\sqrt{d})}$ with t and u positive integers. Put

$$x_k + y_k\sqrt{d} = \alpha\varepsilon^{2k}.$$

Such recurrence sequences are not as well-studied as Lucas sequences (those defined by $y_k = \left(\beta^k - \gamma^k\right)/(\beta-\gamma)$ for $k \geq 0$), but squares in such sequences arise naturally from the study of diophantine equations

$$X^2 - dY^4 = e,$$

which are of considerable interest, in part since they are quartic models of elliptic curves.

Our technique is based on the use of classical hypergeometric functions, which have long had important applications for diophantine problems. We initially developed this technique for $b=1$ and $-N_{\mathbb{Q}(\sqrt{d})/\mathbb{Q}}(\alpha)$ either a perfect square or a prime power. In these cases, I was able to obtain results that are best possible. More recently, I have also been able to treat arbitrary values of b and $N_{\mathbb{Q}(\sqrt{d})/\mathbb{Q}}(\alpha) < 0$. Our results in this more general setting are no longer best-possible, but the bounds are small absolute constants, nonetheless.