Effective results on twisted Thue equations

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Thue equations

Let $F \in \mathbb{Z}[X, Y]$ be a homogeneous, irreducible polynomial of degree at least three and $m \in \mathbb{Z}$, with $m \neq 0$, then the Diophantine equation

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is called a Thue equation.
Alternatively, let $K = \mathbb{Q}(\alpha)$ be a number field with $[K : \mathbb{Q}] \geq 3$ and $m \in \mathbb{Z}$, with $m \neq 0$. Then we call the norm form equation

$$N_{K/\mathbb{Q}}(X - \alpha Y) = m$$

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- Thue showed in 1918 that a Thue equation has only finitely many solutions.
- Baker proved in 1968 effective finiteness results for Thue equations.
- Tzanakis and de Weger (1989) implemented practical algorithms to solve Thue equations.
- Bugeaud and Győry (1996) gave explicit bounds for the solutions.

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In particular Mignotte (1993) showed that $|t| \geq 4$ is sufficiently large.

Exponentially parameterized Thue equations

Let $(G_n^{(0)}), \ldots, (G_n^{(d)})$ be linear recurrence sequences defined over the integers. Then we consider the family of Thue equations

$$G_n^{(0)} X^d + G_n^{(1)} X^{d-1} Y + \cdots + G_n^{(d)} Y^d = m.$$

We want to find all solutions $(X, Y; n) \in \mathbb{Z}^2 \times \mathbb{N}$.

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E.g. Hilgart (2021) proved that if (A_n) and (B_n) satisfy some mild technical conditions, then the exponentially parameterized Thue equation

$$(X - A_n Y)(X - B_n Y)X - Y^3 = \pm 1$$

has only solutions with $|Y| \leq 1$ provided that n is large.

Twisted Thue equations

Another form of parameterized Thue equations is to twist them. Let $K = \mathbb{Q}(\alpha)$ and $t \geq 1$ an integer, then we call

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More generally let $K/\mathbb{Q}$ be a number field $\gamma_1, \ldots, \gamma_s \in \mathbb{Z}_K$. Then we call

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Then we call

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a twisted Thue equation in s parameters.
For technical reasons we consider only those solutions $(X, Y; t_1, \dots, t_s)$ such that $K = \mathbb{Q}(\gamma_1^{t_1} \cdots \gamma_s^{t_s})$.

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- Finiteness of solutions (2013). The result is not effective.
- Effective finiteness results for solutions in the case that the γ_i are units and under some size restrictions (2013).
- Effective finiteness results for solutions in the one parameter case (2017).

Main result

Theorem (Hilgart, Z.)

Let K be a number field of degree $d \geq 3$ and $s \leq d-2$. Let $\gamma_1, \ldots, \gamma_s \in K^*$ be multiplicatively independent algebraic integers such that for each choice of $d-1$ embeddings $\tilde{\sigma}_1, \ldots, \tilde{\sigma}_{d-1} \in \mathrm{Hom}_{\mathbb{Q}}(K, \mathbb{C})$, we have

$$\operatorname{rank}\begin{pmatrix} \log\left|\frac{\tilde{\sigma}_1(\gamma_1)}{\tilde{\sigma}_{d-1}(\gamma_1)}\right| & \cdots & \log\left|\frac{\tilde{\sigma}_1(\gamma_s)}{\tilde{\sigma}_{d-1}(\gamma_s)}\right| \\ \vdots & \ddots & \vdots \\ \log\left|\frac{\tilde{\sigma}_{d-2}(\gamma_1)}{\tilde{\sigma}_{d-1}(\gamma_1)}\right| & \cdots & \log\left|\frac{\tilde{\sigma}_{d-2}(\gamma_s)}{\tilde{\sigma}_{d-1}(\gamma_s)}\right| \end{pmatrix} = s. \qquad (*)$$

Then the Thue equation

$$\left|N_{K/\mathbb{Q}}\left(X - \gamma_1^{t_1} \cdots \gamma_s^{t_s} Y\right)\right| = 1 \qquad (1)$$

has only finitely many integer solutions $(X, Y, (t_1, \ldots, t_s)) \in \mathbb{Z}^2 \times \mathbb{N}^s$, where $XY \neq 0$ and $\mathbb{Q}\left(\gamma_1^{t_1} \cdots \gamma_s^{t_s}\right) = K$.

Rank condition and Schanuel's conjecture

Conjecture (Schanuel's conjecture)

Given any n complex numbers $z_1, \ldots, z_n$ that are linearly independent over the rational numbers $\mathbb{Q}$, the field extension $\mathbb{Q}(z_1, \ldots, z_n, e^{z_1}, \ldots, e^{z_n})$ has transcendence degree at least n over $\mathbb{Q}$.

If Schanuel's conjecture holds, then the multiplicative independence of $\gamma_1, \ldots, \gamma_s \in K^*$ implies the rank condition $(*)$.

How to solve Thue equations – Part I

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$$F(X, Y) = (X - \alpha_1 Y) \cdots (X - \alpha_d Y).$$

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Let $(X, Y) \in \mathbb{Z}^2$ be a solution then set $\beta_i = X - \alpha_i Y$. Let us choose the index j such that $|\beta_j|$ is minimal then we have

$$|\beta_j| \ll \frac{1}{|Y|^{d-1} \prod_{i \neq j} |\alpha_j - \alpha_i|}$$

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Siegel's identity for distinct indices k,l,j is

$$\beta_j(\alpha_k-\alpha_l)+\beta_k(\alpha_l-\alpha_j)+\beta_l(\alpha_j-\alpha_k)=0.$$

How to solve Thue equations – Part I

We get

$$\underbrace{\frac{\beta_j}{\beta_k}\cdot\frac{\alpha_k-\alpha_l}{\alpha_j-\alpha_l}}_{=:L}+\underbrace{\frac{\beta_l}{\beta_k}\cdot\frac{\alpha_j-\alpha_k}{\alpha_j-\alpha_l}}_{=:L'}=1.$$

and get

$$\log|L'| \ll |Y|^{-d}.$$

Since β_l and β_k are units we can write them as a product of fundamental units $\eta_1,\dots,\eta_t$ in the normal closure of $K=\mathbb{Q}(\alpha_1)$ and obtain an inequality of the form

$$\left|b_1\log|\eta_1|+\dots+b_t\log|\eta_t|+\log\left|\frac{\alpha_j-\alpha_k}{\alpha_j-\alpha_l}\right|\right| \ll |Y|^{-d}$$

An application of Baker-type bounds we obtain

$$\log\log|Y| \gg \log h(\beta_j) \gg \log\max\{|b_i|\} \gg \log|Y|,$$

which yields a contradiction for large $|Y|$.

The theorem of Bugeaud and Győry

Theorem (Bugeaud, Győry 1996)

Let $B \geq \max(|m|, e)$*,* f *be an irreducible polynomial with root* α *and* $K = \mathbb{Q}(\alpha)$*. Let* R *be the regulator of* K *and* r *be the unit rank. Let* H *be an upper bound to the absolute values of the coefficients of* f *and* $n = \deg f \geq 3$*. Let* $F(X, Y) = Y^n f\left(\frac{X}{Y}\right)$*, then all solutions* $(X, Y) \in \mathbb{Z}^2$ *of the Thue equation* $F(X, Y) = m$ *satisfy*

$$\log \max(|X|, |Y|) \leq c \cdot R \cdot \max(\log R, 1)(R + \log(HB)),$$

where $c = 3^{r+27}(r+1)^{7r+19} n^{2n+6r+14}$.

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After reshuffling the indices, we can further assume that $|\sigma_1| \geq \cdots \geq |\sigma_d|$.

A gap principal for S-units

To avoid the first problem we use a gap-principal for S-units which was proved in similar forms by Tijdeman (1973) and Stewart (2018).

Lemma (HZ 2022)

Let K be a number field of degree $d \geq s$ and $\gamma_1, \dots, \gamma_s \in K^*$ multiplicatively independent. Let $\gamma = \gamma(t_1, \dots, t_s) = \gamma_1^{t_1} \cdots \gamma_s^{t_s}$ for non-zero integers $t_1, \dots, t_s$. Then for any two conjugates $\gamma^{(1)}, \gamma^{(2)}$ of γ with $M = \left|\gamma^{(1)}\right| > \left|\gamma^{(2)}\right| = m$ there exists an effectively computable constant c independent of $t_1, \dots, t_s$ such that

$$M - m > \frac{M}{h(M)^c}.$$

Application of the gap principal

We apply this lemma to the product in the inequality

$$|\beta_j| \ll \frac{1}{|Y|^{d-1} \prod_{\substack{i\in\{1,\dots,d\} \\ i\neq j}} |\sigma_j - \sigma_i|}$$

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and get (after some computations):

$$L := \frac{\beta_j}{\beta_k} \cdot \frac{\sigma_k - \sigma_l}{\sigma_j - \sigma_l} \ll \frac{\log|\sigma_1|^{(d-1)c}}{|Y|^d |\sigma_1|^{\frac{1}{d-1}}} \cdot \frac{|\sigma_k - \sigma_l|}{\left|\sigma_j - \sigma_k\right| \left|\sigma_j - \sigma_l\right|}.$$

Two cases

We distinguish now between two cases:

Case 1 There exist at least two distinct indices $i \in \{1, \dots, d\} \setminus \{j\}$ such that $\left|\log\left|\frac{\sigma_i}{\sigma_j}\right|\right| \geq \kappa \log t$, where κ is a (fixed but) sufficiently large constant independent of the solution $(X, Y, (t_1, \dots, t_s))$.

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Case 2 For all but one index $i \in \{1, \ldots, d\} \setminus \{j\}$, we have $\log\left|\frac{\sigma_i}{\sigma_j}\right| \leq \kappa \log t$.

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$$|\sigma_1 - \sigma_l| = |\sigma_1|\left|1 - \frac{\sigma_l}{\sigma_1}\right| \gg |\sigma_1|.$$

This, yields for some positive, effectively computable constant c_1

$$L \ll \frac{\log|\sigma_1|^{(d-1)c}}{|Y|^d |\sigma_1|^{1+\frac{1}{d-1}}} \ll e^{-c_1 t}.$$

Case 1 – Construction of a linear form I

We now return to Siegel's identity $L + L' = 1$ and get

$$\left|\log L'\right| = \left|\log\left|\frac{\beta_l}{\beta_k}\right| + \log\left|\frac{\sigma_j - \sigma_k}{\sigma_j - \sigma_l}\right|\right| = \left|\log\left|1 - L\right|\right| \ll e^{-c_1 t}.$$

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Let us write $\sigma_A = \max\left(\left|\sigma_j\right|, \left|\sigma_k\right|\right)$, $\sigma_a = \min\left(\left|\sigma_j\right|, \left|\sigma_k\right|\right)$ and $\sigma_B = \max\left(\left|\sigma_j\right|, \left|\sigma_l\right|\right)$, $\sigma_b = \min\left(\left|\sigma_j\right|, \left|\sigma_l\right|\right)$. Then

$$\log\left|\frac{\sigma_j - \sigma_k}{\sigma_j - \sigma_l}\right| = \log\frac{\sigma_A}{\sigma_B} + \log\left|\frac{1 - \sigma_a/\sigma_A}{1 - \sigma_b/\sigma_B}\right|.$$

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$$\left|\log L'\right| = \left|\log\left|\frac{\beta_l}{\beta_k}\right| + \log\left|\frac{\sigma_j - \sigma_k}{\sigma_j - \sigma_l}\right|\right| = |\log|1 - L|| \ll e^{-c_1 t}.$$

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Since k, l satisfy $\frac{\sigma_a}{\sigma_A}, \frac{\sigma_b}{\sigma_B} \le t^{-\kappa}$ we get

$$\log\left|\frac{1 - \sigma_a/\sigma_A}{1 - \sigma_b/\sigma_B}\right| = O\left(t^{-\kappa}\right),$$

hence

$$\Lambda = \left|\log\left|\frac{\beta_l}{\beta_k}\right| + \log\frac{\sigma_A}{\sigma_B}\right| \ll t^{-\kappa}.$$

Case 1 – Construction of a linear form II

Assume for now that $\Lambda \neq 0$, r the unit rank of K, then we have

$$\beta_k = \left(\eta_1^{(k)}\right)^{b_1} \cdots \left(\eta_r^{(k)}\right)^{b_r}$$

in terms of the fundamental units $\eta_1, \ldots, \eta_r$ (and similar for β_l).

Case 1 – Construction of a linear form II

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in terms of the fundamental units $\eta_1, \ldots, \eta_r$ (and similar for β_l). We can write $\sigma_A = \left(\gamma_1^{(A)}\right)^{t_1} \cdots \left(\gamma_s^{(A)}\right)^{t_s}$, same for σ_B. We can thus write

$$\Lambda = \left| \sum_{i=1}^{r} b_i \left(\log \left| \eta_i^{(l)} \right| - \log \left| \eta_i^{(k)} \right| \right) + \sum_{i=1}^{s} t_i \left(\log \left| \gamma_i^{(A)} \right| - \log \left| \gamma_i^{(B)} \right| \right) \right|$$
$$\ll t^{-\kappa}.$$

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Therefore we have

$$\left|\sum_{n=1}^{r} b_n \left(\log\left|\eta_n^{(i)}\right|\right)\right| = |\log|\beta_i||$$

$$= |\log(X - \sigma_i Y)| \ll \log|Y| + \log|\sigma_1| \ll t.$$

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We get $\max\{|b_i|\} \ll t$. But applying lower bounds for linear forms in logarithms we obtain

$$\kappa \log t \leq c_2 \log t$$

which yields a contradiction, if we choose κ large enough.

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$$\underbrace{\begin{pmatrix} \log\left|\frac{\gamma_1^{(1)}}{\gamma_1^{(d)}}\right| & \cdots & \log\left|\frac{\gamma_s^{(1)}}{\gamma_s^{(d)}}\right| \\ \vdots & \ddots & \vdots \\ \log\left|\frac{\gamma_1^{(d-2)}}{\gamma_1^{(d)}}\right| & \cdots & \log\left|\frac{\gamma_s^{(d-2)}}{\gamma_s^{(d)}}\right| \end{pmatrix}}_{=\Gamma} \begin{pmatrix} t_1 \\ \vdots \\ t_s \end{pmatrix} \ll \begin{pmatrix} \log t \\ \vdots \\ \log t \end{pmatrix}$$

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holds.

This yields $t \ll \log t$, provided that Γ has full rank.

Outlook – A conjecture of Levesque and Waldschmidt I

Consider the family of simplest cubic fields $K_n = \mathbb{Q}(\alpha_n)$, where the minimal polynomial of $\alpha = \alpha_n$ is

$$X^3 - (n-1)X^2 - (n+2)X - 1.$$

It is well known that $\epsilon = \alpha$ and $\delta = -\frac{1}{\alpha+1}$ are multiplicative independent units. Levesque and Waldschmidt conjecture that the twisted Thue equation

$$N_{K_n/\mathbb{Q}}(X - \epsilon^s \delta^t Y) = \pm 1$$

has at most finitely many solutions (X, Y, n, s, t) with $\max\{|X|, |Y|\} \geq 2$ and $(s, t) \neq (0, 0)$.

Outlook – A conjecture of Levesque and Waldschmidt II

Let us write $\sigma_{s,t} = \epsilon^s \delta^t$. Then it seems that our method is applicable to the conjecture of Levesque and Waldschmidt as long as not all conjugates of $\sigma_{s,t}$ are close together.

Outlook – A conjecture of Levesque and Waldschmidt II

Let us write $\sigma_{s,t} = \epsilon^s \delta^t$. Then it seems that our method is applicable to the conjecture of Levesque and Waldschmidt as long as not all conjugates of $\sigma_{s,t}$ are close together. This occurs if $2s \simeq t$ or $s \simeq 2t$ or $-s \simeq t$.

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Problem

Find all solutions (X, Y, n, T, S) *with* $\max\{|X|, |Y|\} \geq 2$ *and* $|S| \ll \log |T|$ *to the parameterized, twisted Thue equations*

$$N_{K_n/\mathbb{Q}}(X - (\epsilon^2 \delta)^T \epsilon^S Y) = \pm 1,$$

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Thank you

Thank you for your attention!