University of Debrecen

Algebra and Number Theory Seminar

# On the Fekete polynomials of principal Dirichlet characters

(joint with S. Chidambaram
J. Minac
N. D. Tân )

# 1. Motivations

$\chi : (\mathbb{Z}/D)^{\times} \to \mathbb{C}^{*}$ Dirichlet character

$\chi : \mathbb{Z} \to \mathbb{C}^{*}$

$$\chi(a) = \begin{cases} \chi(a \bmod D) & \text{if } \gcd(a,D)=1 \\ 0 & \text{if } \gcd(a,D)>1 \end{cases}$$

$\chi(ab) = \chi(a)\,\chi(b)$

$\chi(a) = \chi(b)$ if $a \equiv b \ (D)$

The L-function of $\chi$

$$L(\chi,s) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}$$

- $L(\chi,s)$ converges absolutely if $\mathrm{Re}(s)>1$
- $L(\chi,s)$ | meromorphic cont to $\mathbb{C}^{*}$
  | holomorphic if $\chi$ is not principal
- Special values of $L(\chi,s)$ contains important info
  $s = 0, -1, -2, \ldots$

__Prop__

$$\Gamma(s)\, L(s,\chi) = \int_0^1 \frac{(-\log(t))^{s-1}}{t} \frac{F_\chi(t)}{1-t^D}\, dt$$

Gamma factor

Where

$$F_\chi(t) = \sum_{a=1}^{D} \chi(a)\, t^a$$

- For example $s=1$

$$L(1,\chi) = \int_0^1 \frac{F_\chi(t)}{t(1-t^D)}\, dt$$

- $\chi = \chi_4$

$$\chi_4(a) = \begin{cases} 0 & \text{if } a \text{ even} \\ 1 & \text{if } a \equiv 1\ (4) \\ -1 & \text{if } a \equiv 3\ (4) \end{cases}$$

$\cancel{1}$ $\cancel{2}$ 3 $\cancel{4}$

0 0

$$F_{\chi_4}(t) = t - t^3$$

$$L(1,\chi_4) = \int_0^1 \frac{t-t^3}{t(1-t^4)}\, dt = \int_0^1 \frac{1}{1+t^2}\, dt$$

$$L(n,\chi_4) = \ldots$$

$$\Gamma(s)\, L(\chi,s) = \int_0^1 \frac{(-\log(t))^{s-1}}{t} \frac{F_\chi(t)}{1-t^D}\, dt$$

- $\chi$ is real. If $F_\chi(t)$ has no zeros on $(0,1)$

$L(\chi,s)$ has no real zeros on $(0,1)$ / Siegel zeros

- $\chi = \chi_p$

$$\chi_p(a) = \left(\frac{a}{p}\right) = \begin{cases} 0 & \text{if } p \mid a \\ 1 & \text{if } a \text{ is a square} \\ -1 & \text{——— not ———} \end{cases}$$

Fekete conjectured that $F_{\chi_p}(t)$ has no zeros on $(0,1)$

- George Polya found a counter example

is 1919, $p = 67$

4. - <u>Distributions of zeros</u> (Conrey, Poonen, Granville, Soundararajan)

- Mahler measure, extremal properties. (Borwein, Choi, Yazdani)

Our works focus on the arithmetic side

$\chi = \chi_p$ (joint with Minac - Tan)

→ $\chi = \chi_\Delta$ quadratic, Minac - Tan

, cubic

→ $\chi = \chi_n$ principal, Chidambaram, Minac, Tan

→ cubic, quartic (in progress)

$\mathbb{Z}[\omega_3]$ $\mathbb{Z}[i]$

$\left(\frac{a}{\pi}\right)_3$

Principal Dirichlet characters

$$\chi_n(a) = \begin{cases} 1 & \text{if } \gcd(a,n)=1 \\ 0 & \text{else} \end{cases}$$

$$\chi : (\mathbb{Z}/n)^\times \to \mathbb{C}^*$$

$$F_n(x) = F_{\chi_n}(x) = \sum_{\substack{a=1 \\ \gcd(a,n)=1}}^{n} x^a$$

Example $n = 3$

$$F_3(x) = x + x^2 = x(1+x)$$

$n = 5$

$$F_5(x) = x + x^2 + x^3 + x^4$$
$$= x(1 + x + x^2 + x^3)$$
$$= x \, \frac{1-x^4}{1-x}$$

$n$: prime

$$F_p(x) = x \, \frac{1-x^{p-1}}{1-x}$$

$$x^m - 1 = \prod_{d|m} \Phi_d(x)$$

$$= x \prod_{\substack{d | p-1 \\ d \neq 1}} \Phi_d(x)$$

$n = 15$

$$F_{15}(x) = x\,\Phi_2(x)\,\Phi_4(x)\,\Phi_8(x)\,f_{15}(x)$$

$$f_{15}(x) = x^6 - x^4 + x^3 - x^2 + 1$$

<u>Numerically</u>

contains arithmetic information

$$F_n(x) = \left(\prod_d \Phi_d^{r_d}\right) \cdot f_n(x)$$

irreducible
maximal Galois group

<u>Question</u> - What $d$ can appear?
- What is $r_d$?

<u>Remark</u> $n_0 = \mathrm{rad}(n)$

$$\gcd(a,n) = 1 \iff \gcd(a, n_0) = 1$$

$$L(\chi_n, s) = L(\chi_{n_0}, s)$$

$$\frac{F_n(x)}{1-x^n} = \frac{F_{n_0}(x)}{1-x^{n_0}}$$

$$\left\{ \begin{array}{c} \text{non-cyc of} \\ F_n \end{array} \right\} \leftrightarrow \left\{ \begin{array}{c} \text{non-cyc factor} \\ \text{of } F_{n_0} \end{array} \right\}$$

We can assume that $n = n_0$, i.e $n$ is a square-free number.

<u>Prop</u> Let $p$ be a prime number, $p \nmid n$

$$\boxed{F_{np}(x) = \frac{1-x^{np}}{1-x^n} F_n(x) - F_n(x^p)}$$

<u>Proof</u>

$$\frac{F_{np}(x)}{1-x^{np}} = \sum_{\gcd(a,np)=1} x^a = \sum_{\gcd(a,n)=1} x^a - \sum_{\gcd(a,n)=1} x^{ap}$$

$$= \frac{F_n(x)}{1-x^n} - \frac{F_n(x^p)}{1-x^{np}}$$

$$F_{np}(x) = \frac{1-x^{np}}{1-x^n} F_n(x) - \underbrace{F_n(x^p)}$$

**Cor** If $d \nmid n$, $p \equiv 1\ (d)$

$\zeta_d$ : primitive $d$-root

. $$\frac{1-\zeta_d^{np}}{1-\zeta_d^{n}} = \frac{1-\zeta_d^{n}}{1-\zeta_d^{n}} = 1$$

. $\zeta_d = \zeta_d^p$

$$F_{np}(\zeta_d) = 0$$

So $\Phi_d$ is a factor of $F_{np}(x)$

$$F_{15}(x) = x\,\Phi_2\,\Phi_4\,\Phi_8\,f_{15}$$

$$15 = 3\times 5$$

Inductively

$$\frac{F_n(x)}{1-x^n} = \sum_{m|n} \underbrace{\mu(m) \frac{x^m}{1-x^m}}$$

Rmk . If $d | n$, Ramanujan sums

$$F_n(\zeta_d) = \frac{\mu(d)\,\varphi(n)}{\varphi(d)} \neq 0$$

If $d|n$ then $\Phi_d$ is never a factor of $F_n$

. Assume that $d \nmid n$

$p, q$ odd prime

$$F_{pq}(x) = \frac{x}{1-x} - \frac{x^p}{1-x^p} - \frac{x^q}{1-x^q} + \frac{x^{pq}}{1-x^{pq}}$$

- $p \equiv 1\ (d)$, $F_{pq}(\zeta_d) = 0$

$\zeta_d\ \zeta_d^{pq} = 1$

$\zeta_d^p\ \zeta_d^q = 1$

$$\zeta_d = \zeta_d^p$$

$$\zeta_d^q = \zeta_d^{pq} = (\zeta_d^q)^p$$

Want

- $d \mid pq+1$

$d \mid p+q$

$\Downarrow$

$\Phi_d$ is a factor

- If $ab = 1$

$$\frac{a}{1-a} + \frac{b}{1-b} = -1$$

$n = 15 = 3 \times 5$

$8 \mid 3+5$, $8 \mid 3.5+1$

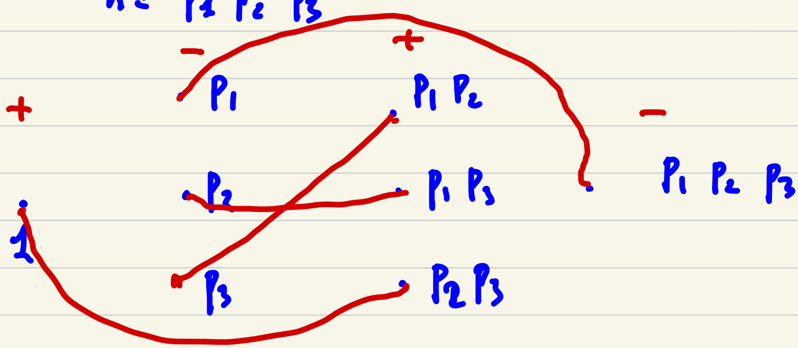
**Thm** Let $N \mid n$

$T$ be a partition of divisors of $N$ into pairs $\{a_i, b_i\}$.

Let $D = \gcd\{\mu(a_i)a_i + \mu(b_i)b_i\}$.

If $d \mid D$ and $d \nmid n$ then $\Phi_d$ is a factor of $F_n$

**Example** $n = p_1 p_2 p_3$




$$\left.\begin{array}{l} d \mid 1 + p_2 p_3 \\ d \mid p_1 p_3 - p_2 \\ d \mid p_1 p_2 - p_3 \\ d \nmid n \end{array}\right\} \quad \Phi_d \text{ is a factor of } F_n$$

<u>Rmk</u> This does <u>NOT</u> catch all $\Phi_d$.

$d$ and $n$ have lots of common factor.