On P-integers and generalizations

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Online Number Theory Seminar
University of Debrecen, Debrecen, Hungary
November 9, 2024

Outline of Talk

Background

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Some useful facts

Some useful lemmas

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More generalizations of Recaman's problem
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- $P^{\star}$-integer

Definition of $P(k)$

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A prime p is called a Recaman prime, if the first p primes form a complete residue system $\pmod{p}$.

Upper bounds of $P(k)$

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Moreover, he made the following conjecture

Conjecture (Chowla, 1934)

$P(k) \ll k^{1+\varepsilon}$, for every $\varepsilon > 0$.

Lower bounds of $P(k)$

Definition

Let $n \geq 1$ be an integer. The Jacobsthal function $j(n)$ is defined as the smallest integer such that any sequence of $j(n)$ consecutive integers contains an element which is coprime to n.

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Theorem (Pomerance, 1980)

Suppose k, m *are integers, with* $0 < m \leq k/(1 + j(k))$ *and* $\gcd(m, k) = 1$. *Then*

$$P(k) > (j(m) - 1)k.$$

Lower bounds of $P(k)$

Theorem (Pomerance, 1980)

For all k, we have

$$P(k) \gg (e^{\gamma} + o(1))\varphi(k)\log k,$$

where $\gamma = 0.577...$ is the Euler's constant.

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In fact, he proved that

$$P(k) > (1 - 4\epsilon)e^{\gamma}\varphi(k)\log k,$$

with $\epsilon > 0$ being arbitrary small and $\varphi(k)$ is the usual Euler φ-function or totient

Pomerance's closing remarks

Question (Racaman, 1978)

Show that there are only finitely many primes p for which the first p primes form a complete residue system modulo p.

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Pomerance generalized this question with

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Show that there are only finitely many positive integers k such that the first $\varphi(k)$ primes which do not divide k form a complete residue system modulo k.

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If k is a P-integer, then $k \leq 30$.

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If k is a P-integer, then $k \leq 30$.

One can easily check that the integers $2, 4, 6, 12, 18, 30$ are P -integers.

Progress toward the proof of Pomerance's conjecture

1. No prime is a P-integer except 2. [Hajdu-Saradha, 2011]

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2. Let $k = 2^{\alpha} k_1 > 1$ with $k_1 = 1$ or $\ell(k_1) > 0.88 \log k$. Then k is a P-integer if and only if $k \in \{2, 4, 6, 12, 18, 30\}$, where $\ell(k)$ denotes the least prime divisor of k with $\ell(1) = 1$. [Saradha, 2011]

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3. All odd primorials are not P-integers. [Saradha, 2011]
   A primorial is a number of the form $N_h = p_1 \cdots p_h$, i.e., the product of the first h primes.

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4. If k is a P-integer with $k > 30$, then $10^{11} < k < 10^{3500}$. [Hajdu-Saradha-Tijdeman, 2012]

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4. If k is a P-integer with $k > 30$, then $10^{11} < k < 10^{3500}$. [Hajdu-Saradha-Tijdeman, 2012]

5. Suppose Riemann Hypothesis holds, then the only P-integers are $k \leq 30$.
   [Hajdu-Saradha-Tijdeman, 2012]

Our main result

Theorem (Yang-T.)

If k is a P-integer, then $k \in \{2, 4, 6, 12, 18, 30\}$.

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Some important facts

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Some important facts

1. From the main result of Hajdu-Saradha-Tijdeman (2012), we suppose that k is a P-integer with $10^{11} < k < 10^{3500}$.

2. Furthermore, due to results from due to Hajdu - Saradha, we may also assume that neither k nor $\frac{k}{2}$ is prime.

3. Let $T = \varphi(k) + \omega(k)$, then there are exactly $\varphi(k)$ primes belonging to the set $\{p_1, \cdots, p_T\}$, which are coprime to k and form a reduced residue system modulo k. The remaining $\omega(k)$ primes in this set divide k.

Key Lemma

Lemma

If an integer k satisfies

$$\sum_{j=0}^{L}\left(2\,\pi\left(\left(j+\frac{1}{2}\right)k\right)-\pi(jk)-\pi\left((j+1)k\right)\right)-1030>0,$$

then k is not a P-integer.

Definition of the Chebyshev function

Definition

The Chebyshev function $\theta(x)$ is defined by .

$$\theta(x) = \sum_{p \leq x} \log p.$$

First lemma

Lemma (Dusart, 2010)

For any $x \in \mathbb{R}$, we have

$$\pi(x) > \frac{x}{\log x} + \frac{x}{\log^2 x} + \frac{2x}{\log^3 x}, \quad \textit{for } x \geq 88783,$$

First lemma

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and

$$\pi(x) < \frac{x}{\log x} + \frac{x}{\log^2 x} + \frac{2.334x}{\log^3 x}, \quad \textit{for } x \geq 2953652287.$$

Second lemma

Lemma (Dusart, 2010)

For any $x \in \mathbb{R}$, *we have*

$$|\theta(x) - x| < \eta_i \frac{x}{\log^i x}, \quad \textit{for } x \geq x_i$$

Second lemma

Lemma (Dusart, 2010)

For any $x \in \mathbb{R}$, *we have*

$$|\theta(x) - x| < \eta_i \frac{x}{\log^i x}, \quad \textit{for } x \geq x_i$$

with $i = 2, \eta_2 = 0.01, x_2 = 7713133853$ *and*
$i = 3, \eta_3 = 0.78, x_3 = 158822621.$

Third lemma

Lemma (Hajdu-Saradha-Tijdeman, 2012)

If k is a P-integer, and let t be an integer such that $tk < p_T < (t+1)k$, then

$$L < L_0 := [\log(k \log k)], \qquad (1)$$

Third lemma

Lemma (Hajdu-Saradha-Tijdeman, 2012)

If k is a P-integer, and let t be an integer such that $tk < p_T < (t+1)k$, then

$$L < L_0 := [\log(k \log k)], \qquad (1)$$

where

$$L = \begin{cases} t-1, & \text{if } p_T \in (tk,\ (t+\frac{1}{2})k), \\ t, & \text{if } p_T \in ((k+\frac{1}{2})k,\ (t+1)k). \end{cases}$$

Definition of $f_j(k)$.

Let integer $k > 10^{11}$. We put

$$f_j(k) = \frac{(2j+1)k}{\log((j+\frac{1}{2})k)}\left(1 - \frac{\eta_i}{\log^i((j+\frac{1}{2})k)}\right) - \frac{jk}{\log(jk)}\left(1 + \frac{\eta_i}{\log^i(jk)}\right) - \frac{(j+1)k}{\log((j+1)k)}\left(1 + \frac{\eta_i}{\log^i((j+1)k)}\right),$$

Definition of $g(k)$.

and

$$g(k) = \frac{k}{\log(\frac{1}{2}k)}\left(1 + \frac{1}{\log(\frac{1}{2}k)} + \frac{2}{\log^2(\frac{1}{2}k)}\right) - \frac{k}{\log k}\left(1 + \frac{1}{\log k} + \frac{2.334}{\log^2 k}\right).$$

Fourth lemma

Lemma

Then, we have

$$\sum_{j=0}^{L}\left(2\pi\left(\left(j+\frac{1}{2}\right)k\right)-\pi(jk)-\pi\left((j+1)k\right)\right)$$

$$> \quad S_L(k) := g(k)+\sum_{j=1}^{L}\left(f_j(k)-\frac{\eta_i\, k}{\log^{i+2}((n+\frac{1}{2})k)}\right),$$

$$\textit{where}\quad \eta_i = \begin{cases} 0.01, & \textit{if}\;\; i=2, \\ 0.78, & \textit{if}\;\; i=3. \end{cases}$$

A sketch of the proof of the conjecture.

When $10^{11} < k \leq 10^{35}$, we take $i = 2$ in Lemma 1. We have

$$\begin{aligned} S_L(k) &= \sum_{j=1}^{L} \Big(\frac{(2j+1)k}{\log((j+\frac{1}{2})k)} \Big(1 - \frac{0.01}{\log^2((j+\frac{1}{2})k)} \Big) \\ &\quad - \frac{jk}{\log(jk)} \Big(1 + \frac{0.01}{\log^2(jk)} \Big) \\ &\quad - \frac{(j+1)k}{\log((j+1)k)} \Big(1 + \frac{0.01}{\log^2((j+1)k)} \Big) - \frac{0.01k}{\log^4(jk)} \Big) \\ &\quad + g(k). \end{aligned}$$

If $10^{11} < k \leq 10^{21}$, then by Lemma 3, we get

$$L \leq L_0 = \lfloor \log(10^{21} \log 10^{21}) \rfloor = 52.$$

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$$u(k, j) = f_j(k) - \frac{0.01k}{\log^4(jk)}.$$

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$$u(k,j) = f_j(k) - \frac{0.01k}{\log^4(jk)}.$$

We prove that $u(k,j)$ ia a stricly decreasing function of j and then that k is not a P-integer.

Similarly, if $10^{21} < k \leq 10^{28}$, we obtain $L_0 = 68$ and $S_L(k) \geq S_{68}(10^{21}) > 3.483 \cdot 10^{16}$.

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If $10^{28} < k \leq 10^{33}$, we get $L_0 = 80$ and $S_L(k) \geq S_{80}(10^{28}) > 9.769 \cdot 10^{21}$.

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If $10^{33} < k \leq 10^{35}$, one has $L_0 = 84$ and $S_L(k) \geq S_{84}(10^{33}) > 1.697 \cdot 10^{28}$.

Similarly, if $10^{21} < k \leq 10^{28}$, we obtain $L_0 = 68$ and $S_L(k) \geq S_{68}(10^{21}) > 3.483 \cdot 10^{16}$.

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If $10^{33} < k \leq 10^{35}$, one has $L_0 = 84$ and $S_L(k) \geq S_{84}(10^{33}) > 1.697 \cdot 10^{28}$.

Hence, when $10^{21} < k \leq 10^{35}$, we deduce that k is not a P-integer.

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We prove that $v(k,j)$ is a strictly decreasing function and that

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Finally, we wrote a simple program in Maple, break down the interval $10^{35} < k < 10^{3500}$ in 25 subintervals and computed T_{L_0}. We realized that the values of T_{L_0} are very high and the conclusion comes from Lemma 1.

Table of the results (Part 1)

$k \in (s_r, s_{r+1}]$	L_0	T_{L_0}
$(10^{35}, 10^{38}]$	91	$1.222 \cdot 10^{30}$
$(10^{38}, 10^{42}]$	101	$2.209 \cdot 10^{32}$
$(10^{42}, 10^{47}]$	112	$1.476 \cdot 10^{36}$
$(10^{47}, 10^{53}]$	126	$4.840 \cdot 10^{40}$
$(10^{53}, 10^{59}]$	140	$4.466 \cdot 10^{47}$
$(10^{59}, 10^{67}]$	159	$5.411 \cdot 10^{52}$
$(10^{67}, 10^{76}]$	180	$1.481 \cdot 10^{61}$
$(10^{76}, 10^{88}]$	207	$4.827 \cdot 10^{68}$

Table of the results (Part 2)

$k \in (s_r, s_{r+1}]$	L_0	T_{L_0}
$(10^{88}, 10^{102}]$	240	$3.850 \cdot 10^{81}$
$(10^{102}, 10^{119}]$	279	$4.659 \cdot 10^{95}$
$(10^{119}, 10^{140}]$	328	$2.617 \cdot 10^{112}$
$(10^{140}, 10^{167}]$	390	$3.487 \cdot 10^{132}$
$(10^{167}, 10^{201}]$	468	$6.112 \cdot 10^{159}$
$(10^{201}, 10^{244}]$	568	$3.339 \cdot 10^{193}$
$(10^{244}, 10^{299}]$	695	$2.941 \cdot 10^{236}$
$(10^{299}, 10^{371}]$	861	$1.557 \cdot 10^{290}$

Table of the results (Part 3)

$k \in (s_r, s_{r+1}]$	L_0	T_{L_0}
$(10^{371}, 10^{463}]$	1073	$3.362 \cdot 10^{363}$
$(10^{463}, 10^{586}]$	1356	$1.135 \cdot 10^{455}$
$(10^{586}, 10^{750}]$	1734	$8.016 \cdot 10^{577}$
$(10^{750}, 10^{974}]$	2250	$5.905 \cdot 10^{740}$
$(10^{974}, 10^{1280}]$	2955	$8.022 \cdot 10^{964}$
$(10^{1280}, 10^{1695}]$	3911	$3.481 \cdot 10^{1271}$
$(10^{1695}, 10^{2280}]$	5258	$1.043 \cdot 10^{1686}$
$(10^{2280}, 10^{3000}]$	6916	$7.392 \cdot 10^{2271}$
$(10^{3000}, 10^{3500}]$	8068	$1.709 \cdot 10^{2992}$

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2. An integer k is called a B-integer, if there exist $\varphi(k)$ *consecutive primes* forming a reduced residue system $\pmod{k}$.
3. A prime k is a shifted P_α-prime if there exist k primes not exceeding $\alpha k \log k$ forming a complete residue system.

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Some examples

Examples

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1. The Recaman prime 2 is also a B-prime as $2, 3$ are two consecutive primes forming a complete residue system $\pmod 2$.

2. One can easily check that the P-integers $2, 4, 6, 12, 18, 30$ are also B-integers.

Lower bounds of $P(k)$

Theorem (Hajdu-Saradha, 2016)

Let k be a prime with the property that there exist k primes not exceeding $\max(p_{\pi(k)+k-1}, 1.1954k\log k)$ which form a complete residue system. Then $k \in \{2, 3, 7, 11\}$.

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In fact, they proved that

$$\max(p_{\pi(k)+k-1}, 1.1954k\log k) = \begin{cases} p_{\pi(k)+k-1}, & \text{if } k < 6691068 \\ 1.1954k\log k, & \text{otherwise.} \end{cases}$$

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The only B-primes are given by $2, 3, 7$.

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In fact, they found the following consecutive primes forming a complete residue system $\pmod 2$, $\pmod 3$, $\pmod 7$ respectively:

$$\{2, 3\}, \quad \{3, 5, 7\}, \quad \{7, 11, 13, 17, 19, 23, 29\}.$$

Second consequence

Theorem (Hajdu-Saradha, 2016)

There is no shifted P_α-prime with $\alpha = 1.1954$.

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One can see that, as $\pi(1.1954k \log k) < k$, then $2, 3, 7$ are not shifted P_α-primes with $\alpha = 1.1954$. This is a consequence of Pomerance second result.

The B-integer conjecture

Conjecture (Hajdu-Saradha, 2016)

Every integer $k \geq 2$ is a B-integer.

$P^{\star}$-integer

An integer k is a P-integer if the block $p_1, p_2, \ldots, p_{\varphi(k)+\omega(k)}$ of the first $\varphi(k) + \omega(k)$ primes, lying in the closed interval $[p_1, p_{\varphi(k)+\omega(k)}]$ has precisely one element in each reduced residue class modulo k, with the exception of $\omega(k)$ primes (which lie in distinct, non-invertible residue classes).

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Remember that $\varphi(k)$ denotes Euler's totient function and $\omega(k)$ the number of distinct prime divisors of k.

$P^{\star}$-integer

Definition

Let $\alpha, \beta, \gamma, \iota > 0$ denote integers, and $G = (G, \cdot)$ an arithmetical semi-group with norm $|\cdot|$, in the sense of Knopfmacher, which takes only values in the positive integers. Consider for $k \in G$ the equivalence relation

$$a \sim b :\Leftrightarrow |a| = |b| \mod |k|$$

on G and let M denote the primes in G with norm in the interval $[\alpha, \beta]$. Then, $k \in G$ is a $P(\alpha, \beta, \gamma, \iota)$-integer or $P^{\star}$-integer if M has in each equivalence class corresponding to an invertible residue class modulo $|k|$ at least γ elements, and the remaining ι primes distribute in some arbitrary equivalence classes such that

$$|M| = \gamma\varphi(k) + \iota.$$

Axiom A

G satisfies Axiom A with $\delta > 0$, if for some $0 \leq \eta < \delta$ the counting function

$$N_G(x) := \#\{g \in G : |g| \leq x\}$$

has the expansion $x^\delta + O(x^\eta)$ as $x \to \infty$.

Finiteness of existence of $P^\star$-integers

Theorem (Elsholtz-Technau-Tichy, 2017)

Let $K := |k|$. Let G be as in the above definition and let G satisfy Axiom A with some $\delta > 0$. Assume that the numbers $\alpha = 1$, $\beta \ll K \log^a K$ and $\iota \ll \log^b K$ are given for some fixed $a, b > 0$ in the case $0 < \delta \leq 1$ and in the case $\delta > 1$ the value of β may additionally differ from multiples of K by at most $K^{1-\epsilon}$, for some absolute constant $\epsilon > 0$. Then, there are only finitely many such $P^\star$-integers.

Remark

If $G = \mathbb{N}$, one sees that the prime counting function is

$$\pi_G(x) := \#\{p \in G : p \textit{ prime}, |p| \leq x\},$$

for $x > 0$.

Extension of Pomerance's result

Theorem (Elsholtz-Technau-Tichy, 2017)

Let $\lambda \in \mathbb{N} \cup \{0\}$ *and* d_1, d_2, d_3 *denote strictly positive real numbers. There are only finitely many* $P(\alpha, \beta, \gamma, \iota)$*-integers in* $\mathbb{N}$ *such that the growth restrictions*

$$\alpha = \lambda k + O(k^{1-d_1}), \quad \iota = O(k^{1-d_2}), \quad \text{and} \quad \beta = O(k \log^{d_3)} k)$$

are satisfied.

Thank you so much!!!
Merci beaucoup!!!