Seqences of Matrices and Substitutions

J. M. Thuswaldner
(with P. Arnoux, V. Berthé, M. Minervino and W. Steiner)

Department of Mathematics and Statistics
University of Leoben
Austria

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Classical continued fractions

- Continued fraction expansion of $x \in \mathbb{R}$

$$x = [a_0; a_1, a_2, a_3, \ldots] = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \ddots}}}.$$

- Convergents

$$\frac{p_n}{q_n} = [a_0; a_1, a_2, a_3, \ldots, a_n].$$

- Recurrence relations for the convergents

$$\begin{aligned} p_n &= a_n p_{n-1} + p_{n-2}, \quad p_{-1} = 1, \quad p_{-2} = 0, \\ q_n &= a_n q_{n-1} + q_{n-2}, \quad q_{-1} = 0, \quad q_{-2} = 1. \end{aligned}$$

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Sequences of matrices

- Let $[a_0; a_1, a_2, \ldots]$ be a continued fraction expansion with convergents

$$\frac{p_n}{q_n} = [a_0; a_1, a_2, a_3, \ldots, a_n].$$

- Then the recurrence relations for p_n and q_n imply

$$\begin{pmatrix} p_n & p_{n-1} \\ q_n & q_{n-1} \end{pmatrix} = \begin{pmatrix} a_0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_n & 1 \\ 1 & 0 \end{pmatrix} \qquad (n \geq 0)$$

- In this sense, the continued fraction algorithm produces a sequence of matrices.

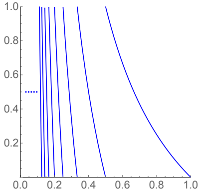
The Gauss map

- For $x \in (0,1)$ the partial quotients $a_1, a_2, \ldots$ of this continued fraction algorithm can be produced by the Gauss map

$$G : (0,1) \mapsto [0,1), \; x \mapsto \frac{1}{x} - \left\lfloor \frac{1}{x} \right\rfloor .$$

- Indeed, one can show that

$$a_n = \left\lfloor \frac{1}{G^{n-1}(x)} \right\rfloor .$$



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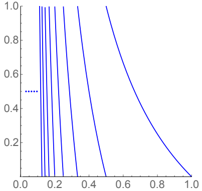

Convergence of a continued fraction algorithm

- A continued fraction algorithm is considered to be "good" if the continued fraction expansion of x converges to x fastly in the sense that

$$\left|\frac{p_n}{q_n} - x\right| \to 0 \qquad (n \to \infty)$$

fastly.

- This is equivalent to the fact that the direction of the sequence of vectors (p_n, q_n) approaches the direction $(x, 1)$ fastly.
- Looking at the matrices this is in turn equivalent to the fact that the Perron-Frobenius eigenvalue of

$$\begin{pmatrix} p_n & p_{n-1} \\ q_n & q_{n-1} \end{pmatrix} = \begin{pmatrix} a_0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_n & 1 \\ 1 & 0 \end{pmatrix}$$

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Approximation of the golden ratio

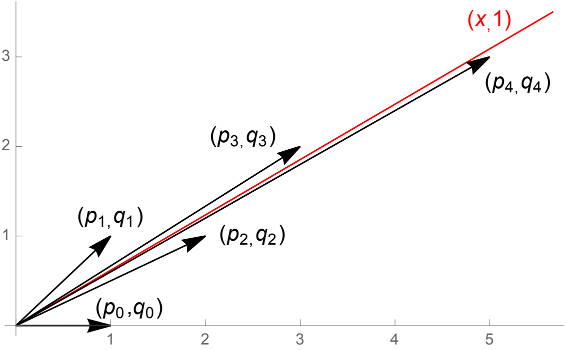


Figure: The approximation of $x = \frac{1+\sqrt{5}}{2}$ by its convergent vectors

$$\bigcap_{n\in\mathbb{N}} \begin{pmatrix} p_n & p_{n-1} \\ q_n & q_{n-1} \end{pmatrix} \mathbb{R}_+^2 \longrightarrow \mathbb{R}_+ \begin{pmatrix} \frac{1+\sqrt{5}}{2} \\ 1 \end{pmatrix} \quad \text{fastly!}$$

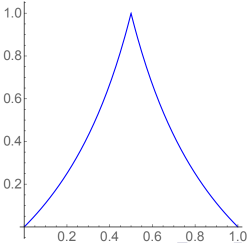
An additive version of this algorithm

- There is a "slower" version of this algorithm.
- it is defined by the map

$$f(x) = \begin{cases} \frac{1-x}{x}, & \text{if } x > \frac{1}{2}, \\ \frac{x}{1-x}, & \text{if } x \leq \frac{1}{2}. \end{cases}$$

- It produces sequences of matrices taken from the set

$$\mathcal{M}_{\text{Additive}} = \left\{ \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \right\}$$



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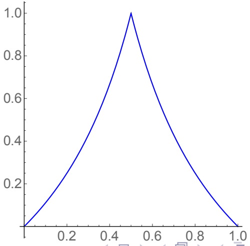

Multidimensional continued fraction algorithms

- This can be generalized to multidimensional continued fraction algorithms
- Multidimensional continued fraction algorithms are used to approximate a vector $(1, x_1, \ldots, x_d)$ by convergents of the form

$$\left(1, \frac{p_{n1}}{q_n}, \ldots, \frac{p_{nd}}{q_n}\right) \sim (q_n, p_{n1}, \ldots, p_{nd}).$$

- Prominent examples are the algorithms of Brun, Selmer, Jacobi-Perron, and Arnoux-Rauzy.
- These algorithms also "produce" sequences of integer matrices (often with determinant ± 1).
- We want to know under which circumstances they provide "good" approximations.
- To this end we study sequences of integer matrices.

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Notations

- Let
$$\mathrm{GL}(d,\mathbb{Z}) = \{M \in \mathbb{Z}^{d\times d} \,:\, \det M = \pm 1\}.$$
- A matrix $M \in \mathbb{Z}^{d\times d}$ with $\det M = \pm 1$ is called unimodular.
- Let
$$\mathcal{M} = \{M \in \mathbb{N}^{d\times d} \,:\, \det M = \pm 1\}.$$
- $\mathbf{M} = (M_n)_{n\in\mathbb{Z}}$ is a sequence of nonnegative matrices in $\mathrm{GL}(d,\mathbb{Z})$.
- For products of consecutive matrices we put
$$M_{[m,n)} = M_m M_{m+1} \cdots M_{n-1} \quad \text{for } m,n \in \mathbb{Z} \text{ with } m \le n,$$
where $M_{[m,m)}$ denotes the $d\times d$ identity matrix.

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Generalized eigenvectors

- Let $\mathbf{M} = (M_n)_{n\in\mathbb{Z}}$ be given.
- $\mathbf{M}$ is primitive (in the future) if for each m there is $n > m$ such that $M_{[m,n)}$ is a positive matrix
- $\mathbf{M}$ is recurrent if each block $M_m, \dots, M_{n-1}$ ($m < n$) occurs infinitely often.

Lemma (Furstenberg:1960)

If a sequence $\mathbf{M} = (M_n)_{n\in\mathbb{Z}}$ of nonnegative matrices in $\mathrm{GL}(d,\mathbb{Z})$ is primitive and recurrent, then there exists a positive vectors $\mathbf{u}$ and $\mathbf{v}$ such that

$$\bigcap_{n\in\mathbb{N}} M_{[0,n)}\mathbb{R}_+^d = \mathbb{R}_+\mathbf{u} \quad \text{(gen. right eigenvector)},$$

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Notions of convergence...

Definition (Weak, strong, and exponential convergence)

- Let $\mathbf{M} = (M_n)_{n\in\mathbb{Z}}$ be a sequence of matrices in $\mathrm{GL}(d,\mathbb{Z})$ and let $\mathbf{u} \in \mathbb{R}^d \setminus \{\mathbf{0}\}$. $\mathbf{M}$ is weakly convergent to $\mathbf{u}$ if

$$\lim_{n\to+\infty} \mathrm{d}\left(\frac{M_{[0,n)}\mathbf{e}_i}{\|M_{[0,n)}\mathbf{e}_i\|}, \frac{\mathbf{u}}{\|\mathbf{u}\|}\right) = 0 \quad \text{for all } i \in \{1,\ldots,d\}.$$

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- $\mathbf{M}$ is exponentially convergent to $\mathbf{u}$ if there exist $C, \alpha > 0$ such that

$$\mathrm{d}(M_{[0,n)}\mathbf{e}_i, \mathbb{R}\mathbf{u}) \le Ce^{-\alpha n} \quad \text{for all } n \in \mathbb{N},\ i \in \{1,\ldots,d\}.$$

... illustrated by a picture

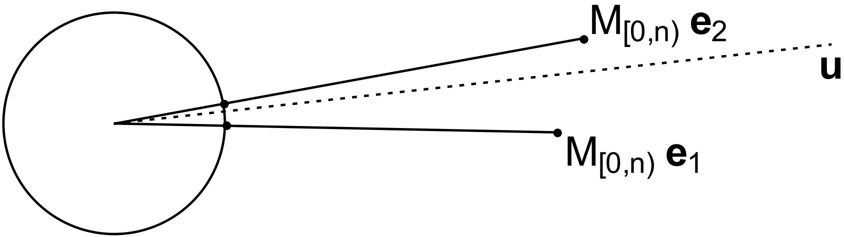
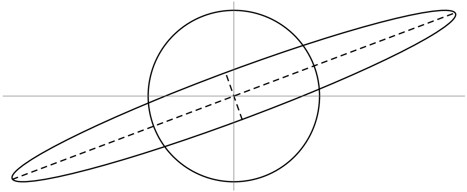


Figure: Weak vs. strong convergence

- Weak convergence takes place on the unit circle.
- Weak convergence means that there is a generalized right eigenvector $\mathbf{u}$.
- Strong convergence takes place at the points $M_{[0,n)}\mathbf{e}_i$.

The Pisot condition

For a matrix $M \in \mathrm{GL}(d, \mathbb{Z})$, $d \geq 2$, the singular values $\delta_1(M), \ldots, \delta_d(M)$ are the eigenvalues of the matrix $({}^tMM)^{1/2}$ assume that $\delta_1(M) \geq \delta_2(M) \geq \cdots \geq \delta_d(M)$.



Definition (Local Pisot condition)

We say that a sequence $\mathbf{M} = (M_n)_{n\in\mathbb{Z}}$ of nonnegative matrices in $\mathrm{GL}(d, \mathbb{Z})$ satisfies the local Pisot condition if

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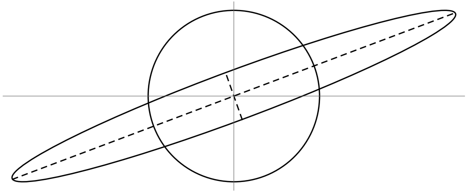

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Algebraic irreducibility and its consequence

Definition (Algebraic irreducibility)

A sequence $\mathbf{M} = (M_n)_{n\in\mathbb{Z}} \in \mathcal{M}_d^{\mathbb{Z}}$ is called algebraically irreducible if for each $m \in \mathbb{Z}$, there is $n_0 \in \mathbb{N}$ such that the characteristic polynomial of $M_{[m,n)}$ is irreducible for each $n \geq n_0$.

Proposition

If a primitive sequence $\mathbf{M} = (M_n)_{n\in\mathbb{Z}} \in \mathcal{M}_d^{\mathbb{Z}}$ satisfies the local Pisot condition and the growth condition $\lim_{n\to\infty} \frac{1}{n} \log\|M_n\| = 0$, then $\mathbf{M}$ is algebraically irreducible and the coordinates of any generalized right eigenvector $\mathbf{u}$ are rationally independent.

A criterion for exponential convergence

Theorem

For $d \geq 2$, let $\mathbf{M} = (M_n)_{n\in\mathbb{Z}} \in \mathcal{M}_d^{\mathbb{Z}}$ be a primitive sequence of nonnegative unimodular integer matrices satisfying the local Pisot condition and the growth condition $\lim_{n\to\infty} \frac{1}{n} \log\|M_n\| = 0$. Then $\mathbf{M}$ is algebraically irreducible and admits a generalized right eigenvector with rationally independent coordinates to which it converges exponentially.

- The proof is hard and uses a local version of the Oseledets theorem (Ruelle, 1979, Arnold, 1998).
- The local Pisot condition (and primitivity) can be checked for classes of examples.
- This theorem also has a metric counterpart.

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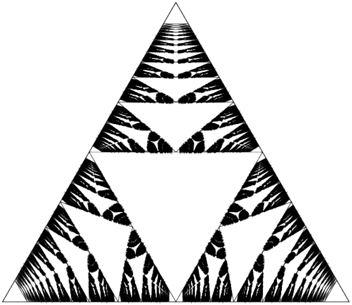
An example: The Arnoux-Rauzy algorithm, I

The Arnoux–Rauzy algorithm (1991) is defined by

$$[1:\alpha:\beta] \mapsto \mathrm{sort}[1-\alpha-\beta:\alpha:\beta] \qquad (1>\alpha>\beta>0)$$

on the set $\{[1:\alpha:\beta] \in \mathbb{P}^2 \,:\, (\alpha,\beta) \in \Delta\}$.



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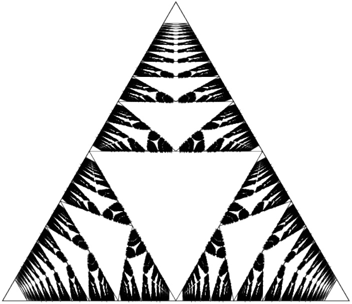

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$$\mathcal{M}_{\mathrm{AR}} = \left\{ \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \right\}.$$

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for some $h \in \mathbb{N}$ and all $n \in \mathbb{Z}$: Strong partial quotients bounded by h.

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- Delecroix, Hejda, and Steiner (2013): There exists a constant $C > 0$, such that for each $n \in \mathbb{N}$, there exists a hyperplane $\mathbf{w}(n)^{\perp}$ such that

$$\|\mathbf{M}_{[0,n)}|_{\mathbf{w}(n)^{\perp}}\| \leq C \left(\frac{2^h - 3}{2^h - 1}\right)^{n/h}.$$

- Hence the hyperplane $\mathbf{w}(n)^{\perp}$ is contracted by $M_{[0,n)}$ in an exponential way, *i.e.*,

$$\limsup_{n\to\infty} \frac{1}{n} \log \|{}^t M_{[0,n)}|_{\mathbf{w}(n)^{\perp}}\| < 0.$$

- Thus by the definition of singular values we have

$$\limsup_{n\to\infty} \frac{1}{n} \log \delta_2(M_{[0,n)}) \leq \limsup_{n\to\infty} \frac{1}{n} \log \|{}^t M_{[0,n)}|_{\mathbf{w}(n)^{\perp}}\| < 0.$$

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$$\limsup_{n\to\infty} \frac{1}{n} \log \|{}^tM_{[0,n)}|_{\mathbf{w}(n)^{\perp}}\| < 0.$$

- Thus by the definition of singular values we have

$$\limsup_{n\to\infty} \frac{1}{n} \log \delta_2(M_{[0,n)}) \leq \limsup_{n\to\infty} \frac{1}{n} \log \|{}^tM_{[0,n)}|_{\mathbf{w}(n)^{\perp}}\| < 0.$$

- Thus the Pisot condition holds.

An example: The Arnoux-Rauzy algorithm, III

- Delecroix, Hejda, and Steiner (2013): There exists a constant $C > 0$, such that for each $n \in \mathbb{N}$, there exists a hyperplane $\mathbf{w}(n)^\perp$ such that

$$\|\mathbf{M}_{[0,n)}|_{\mathbf{w}(n)^\perp}\| \leq C \left(\frac{2^h - 3}{2^h - 1}\right)^{n/h}.$$

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Thus we have the following result, see Berthé, Steiner, and T. (2019):

Theorem (Exponentially convergent Arnoux-Rauzy sequences)

If $\mathbf{M} \in \mathcal{M}_{\mathrm{AR}}$ *is a sequence of* Arnoux-Rauzy matrices *with bounded strong partial quotients then*

- $\mathbf{M}$ *is* algebraically irreducible*.*
- $\mathbf{M}$ *admits a* generalized right eigenvector $\mathbf{u}$ *with rationally independent coordinates.*
- $\mathbf{M}$ *is* exponentially convergent *to* $\mathbf{u}$

There is also a metric theory that allows to prove that almost all (in a certain sense) sequences in $\mathcal{M}_{\mathrm{AR}}$ are exponentially convergent, see again Berthé, Steiner, and T. (2019).

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Markov Partition: Example, I

- Consider the Matrix

$$M = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}.$$

- $\det M = 1$. Thus we can view M as an automorphism on $\mathbb{T}^2$.
- The eigenvalues of M are φ^2, φ^{-2}, where φ is the golden ratio.
- The eigenvectors are given by.

$$\begin{pmatrix} \varphi \\ 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} -1/\varphi \\ 1 \end{pmatrix}$$

- Goal: Give a symbolic representation of $M : \mathbb{T} \to \mathbb{T}$.

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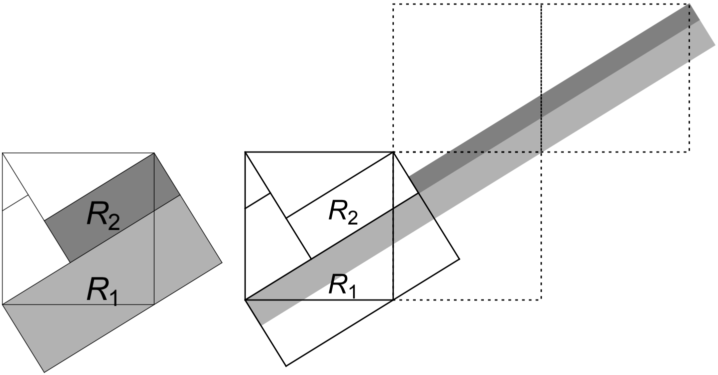

Figure: Using the eingendirections we build an L-shaped fundamental domain of $\mathbb{T}^2$. The boxes R_1 and R_2 are mapped nicely to parallel boxes.

Markov Partition: Example, III

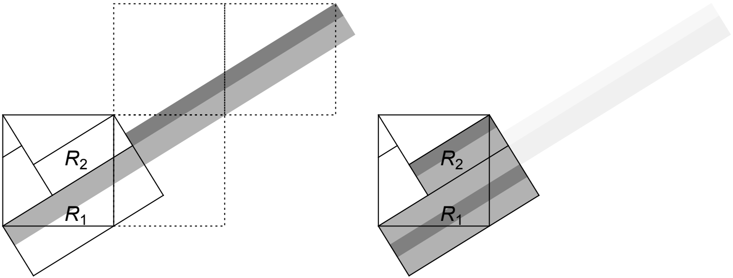

Figure: We can restack the two rectangles to the torus.

- Consider $x \in \mathbb{T}$ and its orbit $(M^n x)_{n \in \mathbb{Z}}$.
- This corresponds “almost” 1:1 to a sequence $(R_{i_k})_{k \in \mathbb{Z}}$.
- M acts as a shift on such a sequence. “Symbolic coding”
- For the general theory see Adler, 1998.

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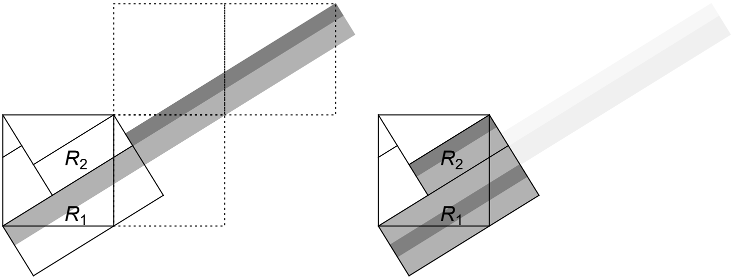

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Markov partitions in higher dimensions

- Consider the matrix

$$M = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

- It has determinant $\det M = 1$, so it can be interpreted as toral automorphism.
- Its eigenvalues satisfy $\lambda_1 > 1 > |\lambda_2| = |\lambda_3|$, so it is hyperbolic.

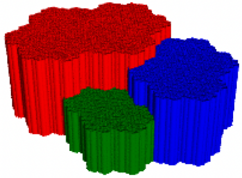

Figure: A Rauzy box and its subtile form a Markov partition for M.

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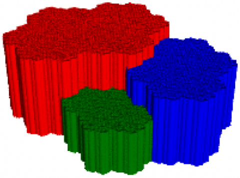

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Restacking

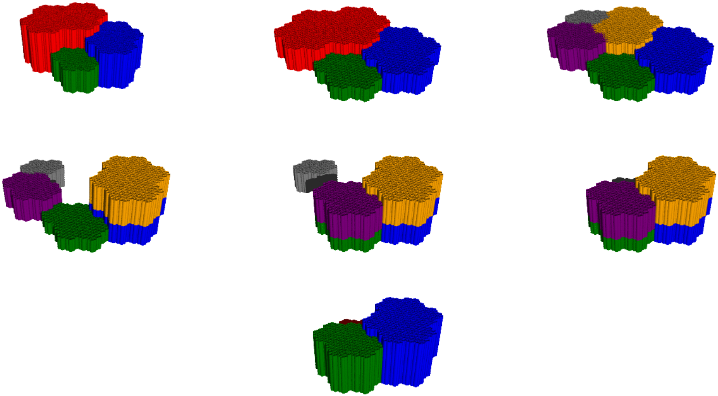

Figure: An illustration of the restacking process for the Tribonacci substitution $\sigma : 1 \mapsto 12, 2 \mapsto 13, 3 \mapsto 1$.

Markov partitions in higher dimensions

- We cannot do this by cuboids in $\mathbb{R}^3$ by Bowen 1978.
- The Rauzy box is defined in terms of substitutions
- Question: Can we define Markov partitions for sequences of matrices by using substitutions as well?
- It turns out that this works.
- In dimension 2 this was done by Arnoux and Fisher, 2001 and 2006 (“scenery flow”). This case is related to the classical continued fraction algorithm.
- In dimension 2 the Markov partitions are again rectangles (although they vary over the sequence of matrices).

Substitutions

- A substitution σ on the alphabet $\mathcal{A}$ is an endomorphism of the free monoid $\mathcal{A}^*$.
- With σ, we associate its incidence matrix

$$M_\sigma = (|\sigma(b)|_a)_{a,b\in\mathcal{A}} \in \mathbb{N}^{d\times d}.$$

- This matrix is the abelianized version of σ in the following sense. Let

$$\mathbf{l} : \mathcal{A}^* \to \mathbb{N}^d, \quad w \mapsto {}^t(|w|_1, \ldots, |w|_d)$$

be the abelianization map, then $\mathbf{l}(\sigma(w)) = M_\sigma \mathbf{l}(w)$ holds for all $w \in \mathcal{A}^*$.

Example: Tribonacci substitution, I

- Let $\mathcal{A} = \{1, 2, 3\}$ be the alphabet.
- Define the tribonacci substitution σ by

$$\sigma(1) = 12, \quad \sigma(2) = 13, \quad \sigma(3) = 1.$$

- For instance, we have $\sigma(1231) = 1213112$.
- Its incidence matrix is given by

$$M_\sigma = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

- We have

$$\begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} = \mathbf{l}(1213112) = \mathbf{l}(\sigma(1231)) = M_\sigma \mathbf{l}(1231) = M_\sigma \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}.$$

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Example: Tribonacci substitution, II

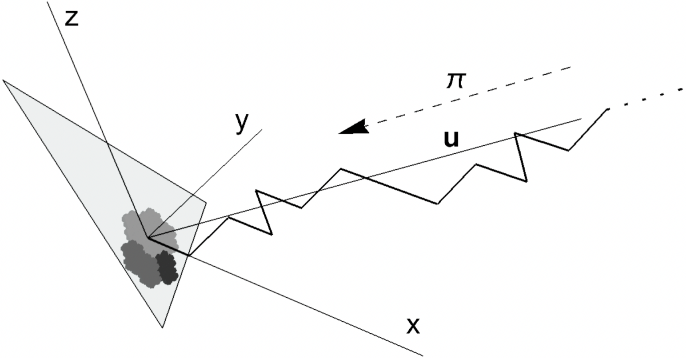

Figure: Projecting a geometric realization of σ yields a Rauzy fractal.

- The "broken line" stays at bounded distance from a vector **u**.
- This is related to a property of σ called balance.

Example: Tribonacci substitution, III

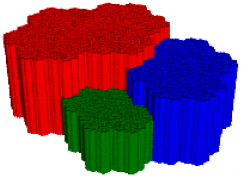

Figure: The Rauzy box again.

- It is no coincidence that the Rauzy fractal and the Markov partition of M_σ look the same.
- More precisely, the Rauzy fractal and its subtiles form the bases of the atoms of the Markov partition of M_σ.

Sequences of substitutions

- Let $\boldsymbol{\sigma} = (\sigma_n)_{n\in\mathbb{Z}}$ be a sequence of substitutions.
- We write

$$\sigma_{[m,n)} = \sigma_m \sigma_{m+1} \cdots \sigma_{n-1} \quad \text{for } m, n \in \mathbb{Z} \text{ with } m \leq n;$$

- The incidence matrix of $\sigma_{[m,n)}$ satisfies

$$M_{\sigma_{[m,n)}} = M_{\sigma_m} M_{\sigma_{m+1}} \cdots M_{\sigma_{n-1}}.$$

- Many properties of $\boldsymbol{\sigma}$ depend only on its sequence of incidence matrices

$$\mathbf{M}_{\boldsymbol{\sigma}} = (M_{\sigma_n})_{n\in\mathbb{Z}}.$$

$\mathcal{S}$-adic shifts

- Let $\boldsymbol{\sigma} = (\sigma_n)_{n \in \mathbb{Z}}$ be a sequence of substitutions.
- Language of $\boldsymbol{\sigma}$:

$$\mathcal{L}_{\boldsymbol{\sigma}}^{(n)} = \{w \in \mathcal{A}^* : w \text{ is a factor of } \sigma_{[n,m)}(a) \text{ for some } a \in \mathcal{A},\ m > n\}$$

- Let $\boldsymbol{\sigma} \in \mathcal{S}^{\mathbb{Z}}$ with $\mathcal{S} \subset \mathcal{S}_d$, $d \geq 2$.

$$X_{\boldsymbol{\sigma}}^{(n)} = \{w \in \mathcal{A}^{\mathbb{Z}} : \text{each factor of } w \text{ is an element of } \mathcal{L}_{\boldsymbol{\sigma}}^{(n)}\}.$$

- The symbolic dynamical system $(X_{\boldsymbol{\sigma}}^{(n)}, \Sigma)$ is called the (two-sided) $\mathcal{S}$-adic shift of level n associated with $\boldsymbol{\sigma}$.
- We let $X_{\boldsymbol{\sigma}} := X_{\boldsymbol{\sigma}}^{(0)}$, and call $(X_{\boldsymbol{\sigma}}, \Sigma)$ the $\mathcal{S}$-adic shift associated with $\boldsymbol{\sigma}$.

Balance

- Many properties like primitivity, recurrence, algebraic irreducibility of σ are inherited from their sequence of incidence matrix.
- Balance makes sense only for substitutions.
- Let $C \in \mathbb{N}$. A set of words $\mathcal{L} \subset \mathcal{A}^*$ is called C-balanced if

$$|v|_a - |w|_a \leq C \quad \text{for all } v, w \in \mathcal{L} \text{ with } |v| = |w| \text{ and for all } a \in \mathcal{A}.$$

- An element of $\mathcal{A}^{\mathbb{Z}}$ is called C-balanced if the set of its factors is C-balanced.
- A shift (X, Σ) is called C-balanced if the set of all factors of its elements is C-balanced.
- If one of these objects is C-balanced for some unspecified C, then we just say that it is balanced.

The Pisot condition implies a lot of things

Here is an analog for our theorem on sequences of matrices.

Theorem

Let $\boldsymbol{\sigma} \in \mathcal{S}^{\mathbb{Z}}$, with $d \geq 2$, and be a primitive sequence of substitutions with sequence of incidence matrices $\mathbf{M}_{\boldsymbol{\sigma}} = (M_{\sigma_n})_{n\in\mathbb{Z}}$. Assume that $\boldsymbol{\sigma}$ satisfies the local Pisot condition *and the growth condition $\lim_{n\to\infty} \cdot \frac{1}{n} \log \|M_{\sigma_n}\| = 0$,*

- *The coordinates of its generalized eigenvector $\mathbf{u}$ are rationally independent and σ converges exponentially and, hence, strongly to $\mathbf{u}$.*
- *The language $\mathcal{L}_{\sigma}$ is balanced.*

Balance is important to define Rauzy fractals!
Again, there is a metric version of this.

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Spectral results, I

We recall some definitions.

- We say that a directive sequence σ has purely discrete spectrum if the system (X_{σ}, Σ) is uniquely ergodic (*i.e.*, it has a unique shift invariant measure μ), minimal, and has purely discrete measure-theoretic spectrum (*i.e.*, the measurable eigenfunctions of the Koopman operator $U_T : L^2(X_{\sigma}, \Sigma, \mu) \to L^2(X_{\sigma}, \Sigma, \mu)$, $f \mapsto f \circ \Sigma$, span $L^2(X_{\sigma}, \Sigma, \mu)$).
- A complex number λ is a continuous eigenvalue of (X, T) if there exists a non-zero continuous function $f : X \to \mathbb{C}$ such that $f \circ T = \lambda f$.

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Lemma (Gottschalck and Hedlund)

Let (X, T, μ) be a minimal and uniquely ergodic subshift of $\mathcal{A}^{\mathbb{Z}}$. If X is balanced on letters, then $\exp(2\pi i \mu([a]))$ is a continuous eigenvalue *of (X, T, μ) for each $a \in \mathcal{A}$.*

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Let $\boldsymbol{\sigma} = (\sigma_n)_{n\in\mathbb{Z}} \in \mathcal{S}_d^{\mathbb{Z}}$, with $d \geq 2$, be a primitive sequence of unimodular substitutions with incidence matrices $(M_{\sigma_n})_{n\in\mathbb{Z}}$ satisfying the Pisot condition and the growth condition $\lim_{n\to\infty} \frac{1}{n} \log \|M_{\sigma_n}\| = 0$. *Then the uniquely ergodic $\mathcal{S}$-adic shift $(X_{\boldsymbol{\sigma}}, \Sigma, \mu)$ is* not weakly mixing. *In particular,* $\exp(2\pi i \mu([a]))$ *is a topological eigenvalue for each letter $a \in \mathcal{A}$. Moreover, the shift admits a* minimal rotation on $\mathbb{T}^{d-1}$ as a topological factor.

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S-adic Rauzy fractals

Definition ($\mathcal{S}$-adic Rauzy fractal)

Let $\boldsymbol{\sigma} = (\sigma_n)_{n\in\mathbb{Z}} \in \mathcal{S}_d^{\mathbb{Z}}$, with $d \geq 2$, be a primitive sequence of unimodular substitutions over the alphabet $\mathcal{A}$ that admits generalized right and left eigenvectors. For $n \in \mathbb{Z}$, the Rauzy fractal $\mathcal{R}_n$ (of level n) associated with $\boldsymbol{\sigma}$ is defined as

$$\mathcal{R}_n = \overline{\{\pi_n \mathbf{l}(p) \,:\, p \preceq \sigma_{[n,m)}(b) \text{ for infinitely many } m \geq n,\, b \in \mathcal{A}\}},$$

and, for each $a \in \mathcal{A}$, a subtile of $\mathcal{R}_n(w)$ is defined as

$$\mathcal{R}_n(a) = \overline{\{\pi_n \mathbf{l}(p) \,:\, p\, a \preceq \sigma_{[n,m)}(b) \text{ for infinitely many } m \geq n,\, b \in \mathcal{A}\}}.$$

Note: For each level n we have a Rauzy fractal.

S-adic Rauzy boxes

Definition (Rauzy box)

Let $\boldsymbol{\sigma} \in \mathcal{S}_d^{\mathbb{Z}}$, with $d \geq 2$, be a primitive sequence of unimodular substitutions over the alphabet $\mathcal{A}$ that admits generalized right and left eigenvectors. For each $n \in \mathbb{Z}$, the Rauzy box (of level n) is

$$\widehat{\mathcal{R}}_n = \bigcup_{a \in \mathcal{A}} \widehat{\mathcal{R}}_n(a),$$

with cylinders $\widehat{\mathcal{R}}_n(a)$ that are defined as the Minkowski sums

$$\widehat{\mathcal{R}}_n(a) = \tilde{\pi}_n \overline{\mathbf{0e}_a} - \mathcal{R}_n(a) \qquad (n \in \mathbb{Z},\ a \in \mathcal{A}). \tag{1}$$

S-adic restacking

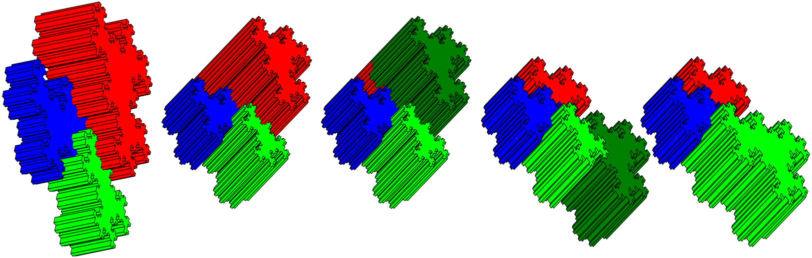

Figure: The general case: When we restack the S-adic Rauzy fractal, we get back the Rauzy fractal of the next level.

- In the S-adic setting we get a nonstationary Markov partition.

A result for sequences of substitutions

Theorem

- *Let $\boldsymbol{\sigma} = (\sigma_n)_{n\in\mathbb{Z}} \in \mathcal{S}_d^{\mathbb{Z}}$, with $d \geq 2$, be a sequence of unimodular substitutions that satisfies the local* Pisot condition *plus a* "coincidence condition"*.*
- *Then the sequence $(\mathcal{P}_n^{\mathrm{gen}})_{n\in\mathbb{Z}}$ defined by subtiles of Rauzy fractals forms a* generating nonstationary Markov partition *for the underlying sequence of matrices $(M_{\sigma_n})_{n\in\mathbb{Z}}$.*

A result for the Brun continued fraction algorithm

Corollary

For $d \in \{3,4\}$, let $(X_{\mathrm{B}}, F_{\mathrm{B}}, A_{\mathrm{B}}, \nu_{\mathrm{B}})$ be the ordered Brun continued fraction algorithm. *Let φ_{B} be the* substitutive realization *of this algorithm that is defined by the ordered Brun substitutions $\sigma_{\mathrm{B},k}$, $k \in \mathcal{A}$. Then, for ν_{B}-almost all $(\mathbf{x}, \mathbf{y}) \in \widehat{X}_{\mathrm{B}}$, the mapping family $(\mathbb{T}, f_{\boldsymbol{\sigma}})$ associated with $\boldsymbol{\sigma}_{\mathrm{B}} = \varphi(\mathbf{x}, \mathbf{y})$ is* eventually Anosov *and admits a* generating nonstationary Markov partition*, whose atoms are explicitly given by Rauzy boxes. This Markov partition provides a symbolic model for $(\mathbb{T}, f_{\boldsymbol{\sigma}})$ as a* nonstationary edge shift.

- Pisot condition Avila and Delecroix, 2019
- Coincidence condition Berthé, Steiner, and T., 2019, 2023