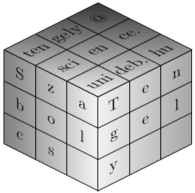
Cubic Diophantine equations related to the Mordell-Schinzel conjecture

Szabolcs Tengely

14 November 2025



Background

There are many interesting results dealing with equations of the form $xyz = G(x, y)$, where $G \in \mathbb{Z}[x, y]$. Already the quadratic case is non-trivial as one may see from the results by Jacobsthal, Mills and Schinzel.
In 1952 Mordell considered the congruence $ax^3 + by^3 + c \equiv 0 \pmod{xy}$ and obtained some related Diophantine results as well. Schinzel extended Mordell's result and considered congruences of the form

$$f(x) + g(y) + c \equiv 0 \pmod{xy},$$

where f, g are polynomials with integral coefficients of degrees at most m and n, respectively.

UNIVERSITY of DEBRECEN

Background

More general cubic equations have a vast literature. A well-studied family is given by

$$x^3 + y^3 + z^3 = n.$$

In 1953 Mordell asked about the integral solutions of the above equation with $n = 3$. Motivated by the question Miller and Woollett did a computer search for solutions with $0 \leq n \leq 100$. There were a few missing values in the given range, for which there were no known solutions. In 2019 Booker solved the case $n = 33$ and in 2021 Booker and Sutherland could handle the equation $x^3 + y^3 + z^3 = 42$.

Result by Kollár and Li

Recently, Kollár and Li obtained a nice result in the topic by applying an elegant argument based on automorphism groups.

Theorem A (Kollár-Li)
The equation

$$xyz = ax^3 + by^3 + c + a_2x^2 + a_1x + b_2y^2 + b_1y$$

has infinitely many integral solutions for every $a, b, c \in \mathbb{Z}/\{0\}$ and $a_1, a_2, b_1, b_2 \in \mathbb{Z}$.

This result follows from Schinzel's work when $|abc| > 1$.

UNIVERSITY of DEBRECEN

Result by Kollár and Li

It remains to consider the equation

$$S_{a_1,a_2,b_1,b_2}: \quad xyz = x^3 + y^3 + 1 + a_2x^2 + a_1x + b_2y^2 + b_1y.$$

The argument of Kollár and Li uses automorphism groups and orbits of integral solutions. In many cases there are trivial solutions and the orbits of those provide infinitely many integral solutions.
We note that equation $S_{0,0,0,0}$ and certain generalizations called generalized Hessian curves have been investigated in cryptography. We remark that equation $S_{0,0,0,0}$ is also related to cyclic cubic number fields.

Result by Kollár and Li

There are 4 special cases, namely

$$\begin{aligned} S_1 : xyz &= x^3+y^3+1-x^2-y^2, \\ S_2 : xyz &= x^3+y^3+1-2x^2-x-2y^2-y, \\ S_3 : xyz &= x^3+y^3+1-2x^2-x-y^2, \\ S_4 : xyz &= x^3+y^3+1-x^2-2y^2-y. \end{aligned}$$

In the above 4 cases a direct search yields non-trivial solutions given by

$$S_1 : (-7,-17,-47), S_2 : (293,-601,1095), S_3 : (11,-13,9), S_4 : (-13,11,9).$$

The corresponding orbits give infinitely many integral solutions.

Result by Kollár and Li - example

Let
$A(x) = x^3 - x^2 + 1, \overline{A}(x) = x^3 - x + 1, B(y) = y^3 - y^2 + 1, \overline{B}(y) = y^3 - y + 1.$
Kollár and Li defined $S_{A,B} : xyz = A(x) + B(y) - 1$ and its companion surfaces given by $S_{\overline{A},B}, S_{A,\overline{B}}, S_{\overline{A},\overline{B}}$. The maps to generate solutions are as follows:

$$\begin{aligned} \sigma_{x,A,B} &: (x, y) \longrightarrow (x, A(x)/y), \\ \sigma_{y,A,B} &: (x, y) \longrightarrow (B(y)/x, y). \end{aligned}$$

The diagram below clockwise gives an infinite order automorphism.

$$\begin{array}{ccc} S_{A,B} & \overset{\sigma_x}{\longleftrightarrow} & S_{A,\bar{B}} \\ \sigma_y \updownarrow & & \updownarrow \sigma_y \\ S_{\bar{A},B} & \overset{\sigma_x}{\longleftrightarrow} & S_{\bar{A},\bar{B}} \end{array}$$

Result by Kollár and Li - example

Generating new solution via the maps:

$$
\begin{aligned}
&S_{A,B} : (-7,-17,-47) \to \sigma_x \to S_{A,\overline{B}} : (-7,23,-73),\\
&S_{A,\overline{B}} : (-7,23,-73) \to \sigma_y \to S_{\overline{A},\overline{B}} : (-1735,23,130879),\\
&S_{\overline{A},\overline{B}} : (-1735,23,130879) \to \sigma_x \to S_{\overline{A},B} : (-1735,-227075593,-29719495771399),\\
&S_{\overline{A},B} : (-1735,-227075593,-29719495771399) \to \sigma_y \to S_{A,B} : (x_1,y_1,z_1).
\end{aligned}
$$

where

$$
\begin{aligned}
x_1 &= 674857212595141735438 3,\\
y_1 &= -227075593,\\
z_1 &= -2005641607601942182309270338791769 47.
\end{aligned}
$$

Cyclic cubic fields

A family of cubic fields studied by Cohn, Ennola and Shanks is related to the parametric polynomial

$$X^3 + (n+3)X^2 + nX - 1 = h_n(X).$$

The family $(h_n(X))n \in \mathbb{N}$ has been further investigated by many authors e.g. by Lecacheux, Lettl and Washington. Kishi and Washington provided other one-parameter families with similar properties. Later, Balady generalized Washington's procedure and described a method for generating other one-parameter families.

Cyclic cubic fields

In this construction polynomials f and g having integral coefficients such that the fraction

$$\frac{f(t)^3 + g(t)^3 + 1}{f(t)g(t)} =: \lambda(t)$$

is also a polynomial with integral coefficients play an important role. More precisely, if $\lambda \in \mathbb{Z}[t]$, then the pair of polynomials (f, g) determines one-parameter family of polynomials

$$P_{f,g}(X) = X^3 + a(n)X^2 + \lambda(n)X - 1,$$

where $a(n) = 3(f(n)^2 + g(n)^2 - f(n)g(n)) - \lambda(n)(f(n) + g(n))$. For all but finitely many values of n, the polynomial $P_{f,g}$ has three real roots which are units and more importantly the field generated by these roots is cyclic cubic field.

Cyclic cubic fields

Ulas and Tengely determined parametric solutions by means of Gröbner basis techniques:

$$\begin{aligned} f(t) &= t, & g(t) &= -t-1, & \lambda(t) &= 3, \\ f(t) &= t, & g(t) &= -t^2+t-1, & \lambda(t) &= t^3-2t^2+3t-3 \\ f(t) &= t, & g(t) &= -t^3-1, & \lambda(t) &= t^2\left(t^3+2\right), \end{aligned}$$

$$f(t) = \frac{1}{2}(t^2-t+1), \quad g(t) = \frac{1}{2}(t^2+t+1) = f(-t), \lambda(t) = t^2+5.$$

$$f(t) = -t^2, \quad g(t) = t^3-1, \lambda(t) = -t(t^3-3).$$

$$f(t) = -t^2+t-1, \quad g(t) = t(t^4-2t^3+4t^2-3t+3).$$

Here we have $\lambda(t) = -t^8+3t^7-8t^6+11t^5-15t^4+10t^3-8t^2-1.$

Recurrence sequences and families of solutions

Tengely and Ulas determined the complete set of integral solutions of equation $S_{0,0,0,0}$. If $z = -t^2$ for some integer t, then there are some parametric solutions with $x, y \in \{-1, 0, \pm t\}$. Let us list the solutions for $3 \le t \le 1000$ with $x, y \notin \{-1, 0, \pm t\}$ only for those values of t for which there are at least 9 solutions.

t	non-trivial solutions (x, y)
5	$[(27,-19),(-19,27),(-2,-7),(-7,-2),(-9,-13),(-13,-9)]$
15	$[(161,-93),(-93,161),(26,-3),(-3,26),(-63,-109),(-109,-63)]$
37	$[(1159,-733),(-733,1159),(-373,-657),(-657,-373),(-437,-691),(-691,-437)]$
99	$[(7791,-4771),(-4771,7791),(-2989,-4881),(-4881,-2989)]$
257	$[(53793,-33361),(-33361,53793),(-20511,-33073),(-33073,-20511)]$
461	$[(27729,-3610),(-3610,27729),(-59,-3541),(-3541,-59)]$
675	$[(367667,-226929),(-226929,367667),(-140529,-227683),(-227683,-140529)]$

Let us now list the divisors that give rise to these solutions.

t	divisors (d_1, d_2)
5	$[(-1, 15652), (-1, 15652), (-52, 301), (-52, 301), (-91, 172), (-91, 172)]$
15	$[(-21, 542412), (-21, 542412), (-156, 73017), (-156, 73017), (-741, 15372), (-741, 15372)]$
37	$[(-91, 28194796), (-91, 28194796), (-4459, 575404), (-4459, 575404), (-4753, 539812), (-4753, 539812)]$
99	$[(-741, 1270553508), (-741, 1270553508), (-33411, 28178748), (-33411, 28178748)]$
257	$[(-4753, 60622092892), (-4753, 60622092892), (-226801, 1270438876), (-226801, 1270438876)]$
461	$[(-140164, 68480838517), (-140164, 68480838517), (-223321, 42980947828), (-223321, 42980947828)]$
675	$[(-33411, 2830956281532), (-33411, 2830956281532), (-1560261, 60621319332), (-1560261, 60621319332)]$

From here one gets that -91 appears at rows of $t = 5$ and of $t = 37$, similarly, -741 appears at rows of $t = 15$ and $t = 99$. It seems to follow a pattern so the sequence $5, 15, 37, 99, \ldots$ is worth being studied. We obtain one closely related sequence in the database OEIS, namely A265762, the only difference is the signs of the numbers. It is described as coefficient of x in minimal polynomial of the continued fraction $[1^n, 2, 1, 1, 1, \ldots]$, where 1^n means n ones. It is a ternary linear recurrence sequence of the form (modified to give appropriate signs)

$$R_0 = 1, \quad R_1 = 3, \quad R_2 = 5,$$
$$R_n = 2R_{n-1} + 2R_{n-2} - R_{n-3}, \quad n \geq 3.$$

The sequence can be written of the form

$$R_n = -\frac{3}{5}(-1)^n + \frac{4}{5}\left(\frac{1}{2}\sqrt{5}+\frac{3}{2}\right)^n + \frac{4}{5}\left(-\frac{1}{2}\sqrt{5}+\frac{3}{2}\right)^n.$$

Also we have the pattern provided by the divisors, this sequence is given by $-1, -21, -91, -741, \dots$. There is no such sequence in OEIS. By means of factorization we have that the sequence is (in absolute value)

$$1\times(2\times1-1), 3\times(2\times3+1), 7\times(2\times7-1), 19\times(2\times19+1), 49\times(2\times49-1), \dots$$

Again we encounter a sequence following the recurrence equation of R_n, namely

$$S_0 = 1, \quad S_1 = 3, \quad S_2 = 7,$$
$$S_n = 2S_{n-1} + 2S_{n-2} - S_{n-3}, \quad n \geq 3.$$

We obtain that

$$S_n = \frac{1}{25}\sqrt{5}\left(\left(\frac{1}{2}\sqrt{5}+\frac{3}{2}\right)^n\left(3\sqrt{5}+5\right)-\sqrt{5}\,(-1)^n+3\sqrt{5}\left(-\frac{1}{2}\sqrt{5}+\frac{3}{2}\right)^n-5\left(-\frac{1}{2}\sqrt{5}+\frac{3}{2}\right)^n\right).$$

Based on the numerical data we may expect to have additional solutions if $z=-R_n^2$ and d_1 is either

$$-S_{n-2}(2S_{n-2}+(-1)^{n+1}) \quad \text{or} \quad -S_n(2S_n+(-1)^{n+1}).$$

We consider the systems of equations

$$\begin{aligned} 3x+3y-R_n^2 &= -S_{n-2}(2S_{n-2}+(-1)^{n+1}),\\ 9x^2-9xy+9y^2+3R_n^2(x+y)+R_n^4 &= \frac{-R_n^6-27}{-S_{n-2}(2S_{n-2}+(-1)^{n+1})} \end{aligned}$$

and

$$\begin{aligned} 3x+3y-R_n^2 &= -S_n(2S_n+(-1)^{n+1}),\\ 9x^2-9xy+9y^2+3R_n^2(x+y)+R_n^4 &= \frac{-R_n^6-27}{-S_n(2S_n+(-1)^{n+1})}. \end{aligned}$$

To simplify the expressions we try to find some common roots of the two ternary linear recurrence sequences R_n and S_n. The next lemma is such a result. We use the sequence of Lucas numbers defined by $L_0 = 2, L_1 = 1$ and $L_n = L_{n-1} + L_{n-2}, n \geq 2$.

Lemma 1

We have the following identities

- $$R_{2k} = \frac{4L_{4k} - 3}{5}, \qquad R_{2k+1} = \frac{4L_{4k+2} + 3}{5},$$
- $$S_{2k} = \frac{2L_{4k+2} - 1}{5}, \qquad S_{2k+1} = \frac{2L_{4k+4} + 1}{5}.$$

UNIVERSITY of DEBRECEN

It remains to resolve the systems of equations in a way that shows that the solutions are integers. Let us consider the first system with even n. That is the system given by

$$\begin{aligned} 3x + 3y - R_{2k}^2 &= -S_{2k-2}(2S_{2k-2} - 1), \\ 9x^2 - 9xy + 9y^2 + 3R_{2k}^2(x+y) + R_{2k}^4 &= \frac{-R_{2k}^6 - 27}{-S_{2k-2}(2S_{2k-2} - 1)}. \end{aligned}$$

Solving the system directly yields very complicated formulas for x so we follow a different idea. Let us introduce the variables $u = L_{4k-2}, v = L_{4k-1}$ and $t = L_{4k}$. Based on the definition of Lucas sequence L_n we have that

$$t = u + v \quad \text{and} \quad v^2 - ut = -5.$$

We work in the polynomial ring $\mathbb{Q}[x, y, u, v, t]$ and we consider the ideal I generated by the polynomials

$$
\begin{aligned}
&t - u - v,\\
&v^2 - ut + 5,\\
&3x + 3y - \left(\frac{4t-3}{5}\right)^2 + \left(\frac{2u-1}{5}\right)\left(2\frac{2u-1}{5} - 1\right),\\
&\left(9x^2 - 9xy + 9y^2 + 3\left(\frac{4t-3}{5}\right)^2 (x+y) + \left(\frac{4t-3}{5}\right)^4\right) \times\\
&\times \left(\frac{2u-1}{5}\right)\left(2\frac{2u-1}{5} - 1\right) - \left(\frac{4t-3}{5}\right)^6 - 27.
\end{aligned}
$$

UNIVERSITY of DEBRECEN

We compute the elimination ideal of I with respect to the variables u, v, y. We end up with a reducible polynomial depending only on two variables x and t. There are two factors that involve both variables

$$64\,t^4 + 48\,t^3 + 200\,t^2x - 324\,t^2 + 200\,tx - 625\,x^2 - 328\,t + 50\,x - 81,$$
$$64\,t^4 + 48\,t^3 + 400\,t^2x - 324\,t^2 + 150\,tx + 625\,x^2 - 78\,t - 650\,x + 669.$$

These are quadratic equations in x, the solutions are as follows

$$\frac{1}{25}\left(2\,t \pm 2\,\sqrt{5\,t^2 - 20} + 1\right)(2\,t + 1),$$
$$\frac{1}{25}\left(-8\,t^2 - 3\,t \pm 5\,\sqrt{5\,t^2 - 20} + 13\right).$$

By the well-known identity we have

$$L_{4k}^2 - 5F_{4k}^2 = 4.$$

Therefore $\pm\sqrt{5t^2-20}$ is $\pm 5F_{4k}$. Having the solutions for x we obtain the solutions for y. To complete this part let us list the (x,y) solutions

$$
\begin{aligned}
(x_1,y_1) &= \left(\frac{1}{25}(2L_{4k}+10F_{4k}+1)(2L_{4k}+1), \frac{1}{25}\left(-8L_{4k}^2-3L_{4k}-25F_{4k}+13\right)\right),\\
(x_2,y_2) &= \left(\frac{1}{25}\left(-8L_{4k}^2-3L_{4k}-25F_{4k}+13\right), \frac{1}{25}(2L_{4k}+10F_{4k}+1)(2L_{4k}+1)\right),\\
(x_3,y_3) &= \left(\frac{1}{25}(2L_{4k}-10F_{4k}+1)(2L_{4k}+1), \frac{1}{25}\left(-8L_{4k}^2-3L_{4k}+25F_{4k}+13\right)\right),\\
(x_4,y_4) &= \left(\frac{1}{25}\left(-8L_{4k}^2-3L_{4k}+25F_{4k}+13\right), \frac{1}{25}(2L_{4k}-10F_{4k}+1)(2L_{4k}+1)\right).
\end{aligned}
$$

In a similar way we study the system of equations in case of odd n, here we have (this time we take the second system first)

$$\begin{aligned} 3x+3y-R_{2k+1}^2 &= -S_{2k+1}(2S_{2k+1}+1), \\ 9x^2-9xy+9y^2+3R_{2k+1}^2(x+y)+R_{2k+1}^4 &= \frac{-R_{2k+1}^6-27}{-S_{2k+1}(2S_{2k+1}+1)}. \end{aligned}$$

This time we use the notation $u=L_{4k+4}, v=L_{4k+3}$ and $t=L_{4k+2}$. Again, we work in the polynomial ring $\mathbb{Q}[x,y,u,v,t]$ and we consider the ideal I generated by the polynomials

$$\begin{aligned} &u-t-v, \\ &v^2-ut+5, \\ &3x+3y-\left(\frac{4t+3}{5}\right)^2+\left(\frac{2u+1}{5}\right)\left(2\frac{2u+1}{5}+1\right), \\ &\left(9x^2-9xy+9y^2+3\left(\frac{4t+3}{5}\right)^2(x+y)+\left(\frac{4t+3}{5}\right)^4\right)\times \\ &\times\left(\frac{2u+1}{5}\right)\left(2\frac{2u+1}{5}+1\right)-\left(\frac{4t+3}{5}\right)^6-27. \end{aligned}$$

Applying the same idea as in the even case we get the solutions, here we only list the final form of the integral solutions (x, y)

$$\begin{aligned}
(x_5, y_5) &= \left(\frac{1}{25}\left(2\,L_{4k+2}+10F_{4k+2}-1\right)\left(2\,L_{4k+2}-1\right), \frac{1}{25}\left(-8\,L_{4k+2}^2+3\,L_{4k+2}+25F_{4k+2}+13\right)\right),\\
(x_6, y_6) &= \left(\frac{1}{25}\left(-8\,L_{4k+2}^2+3\,L_{4k+2}+25F_{4k+2}+13\right), \frac{1}{25}\left(2\,L_{4k+2}+10F_{4k+2}-1\right)\left(2\,L_{4k+2}-1\right)\right),\\
(x_7, y_7) &= \left(\frac{1}{25}\left(2\,L_{4k+2}-10F_{4k+2}-1\right)\left(2\,L_{4k+2}-1\right), \frac{1}{25}\left(-8\,L_{4k+2}^2+3\,L_{4k+2}-25F_{4k+2}+13\right)\right),\\
(x_8, y_8) &= \left(\frac{1}{25}\left(-8\,L_{4k+2}^2+3\,L_{4k+2}-25F_{4k+2}+13\right), \frac{1}{25}\left(2\,L_{4k+2}-10F_{4k+2}-1\right)\left(2\,L_{4k+2}-1\right)\right).
\end{aligned}$$

UNIVERSITY of DEBRECEN

Let us now formulate a statement based on the previous computations.

Theorem 1

The Diophantine equation

$$x^3 + y^3 + 1 - zxy = 0$$

has parametric solutions described by the formulas $(x_i, y_i), i \in \{1, 2, 3, 4\}$ if $z = -R_{2k}^2$ and by the formulas $(x_i, y_i), i \in \{5, 6, 7, 8\}$ if $z = -R_{2k+1}^2$.

Proof of Theorem 1

The formulas were obtained by assuming some properties based on patterns. Still we need to show that the solutions are actually solutions of the Diophantine equation and also these are not only solutions but integral solutions. The latter part is simpler, divisibility by 25 is what we need. The binary linear recurrence sequences F_n and L_n are periodic modulo 25. The periods of F_{4k} and F_{4k+2} modulo 25 are equal to 25 and of L_{4k} and L_{4k+2} equal to 5. It turns out that the formulas yield integers.

Proof of Theorem 1

It remains to show that the 8 formulas yield solutions. This can be done via Gröbner basis computation. As an example let us consider the first formula (x_1, y_1). We work in $\mathbb{Q}[u, v]$, where u is a variable representing F_{4k} and v represents L_{4k}. We consider the ideal I generated by the polynomial $v^2 - 5u^2 - 4$, the well-known identity we mentioned earlier. Let us take the polynomial

$$f_1(u, v) = x_1^3 + y_1^3 + 1 + \left(\frac{4v - 3}{5}\right)^2 x_1 y_1.$$

The reduction of the polynomial f_1 by the Gröbner basis G of ideal I yields 0, hence f_1 is in I. Similarly one can prove that all 8 formulas provide solutions.

Arithmetic progressions

An arithmetic progression related to the solutions of a Diophantine equation $F(x, y) = 0$, is an arithmetic progression in either the x or y coordinates. One can ask the following natural question. What is the longest arithmetic progression in the x coordinates? In case of linear polynomials, Fermat claimed and Euler proved that four distinct squares cannot form an arithmetic progression. Allison determined an infinite family of quadratics containing an integral arithmetic progression of length eight. Arithmetic progressions related to the equation, called Pellian equation, $x^2 - dy^2 = m$ have been considered by many mathematicians.

UNIVERSITY of DEBRECEN

Arithmetic progressions

Dujella, Pethő and Tadić proved that for any four-term arithmetic progression, except $\{0, 1, 2, 3\}$ and $\{-3, -2, -1, 0\}$, there exist infinitely many pairs (d, m) such that the terms of the given progression are y-components of solutions. Pethő and Ziegler studied 5-term progressions on Pellian equations. In 2013 Aguirre, Dujella and Peral computed 6-term arithmetic progression on Pellian equations parametrized by points on certain elliptic curves having positive rank. Bremner described an infinite family of elliptic curve of Weierstrass form with 8 points in arithmetic progression. González-Jiménez proved that these arithmetic progressions cannot be extended to 9 points arithmetic progressions. Bremner, Silverman and Tzanakis considered the congruent number curve $y^2 = x^3 - n^2x$.

Arithmetic progressions

Let us now consider the parametric solutions from arithmetic progression point of view. We have the list of trivial solutions

$$(-1,0),(0,-1),(-1,\pm R_n),(\pm R_n,-1).$$

Definitely it is easy to get a 3-term arithmetic progression, since $-R_n, 0, R_n$ is such an example. Let us define the lists of x-coordinates of the parametric solutions in the even and odd cases

$$\begin{aligned} S_1 &= [-1,0,\pm R_{2k},x_1,x_2,x_3,x_4], \\ S_2 &= [-1,0,\pm R_{2k+1},x_5,x_6,x_7,x_8]. \end{aligned}$$

Theorem 2

If 4 elements of a 5-term arithmetic progression are from S_1, then $k \in \{0,1\}$. There exists no $k \in \mathbb{Z}_{\geq 0}$ for which 4 elements of a 5-term arithmetic progression are from S_2.

Proof of Theorem 2

Let us deal with the statement in case of S_1, the idea of proof works for S_2 too. We work in the polynomial ring $\mathbb{Q}[u, v, a, d]$, where u is a variable representing F_{4k} and v represents L_{4k}. For a given list of 4 elements of S_1, denote it by $\{f_1, f_2, f_3, f_4\}$, and a given permutation of a 4 element subset of $\{a, a+d, a+2d, a+3d, a+4d\}$, denote it by $\{g_1, g_2, g_3, g_4\}$ we consider the ideal $I_{f,g}$ generated by the polynomials $f_i - g_i, i \in \{1, 2, 3, 4\}$ and $v^2 - 5u^2 - 4$. We determine the elimination ideal of $I_{f,g}$ with respect to the variables a, d. In all cases, we get a zero-dimensional ideal and we obtain that v is either 2 or 7. Variable v denotes L_{4k} and $L_0 = 2, L_4 = 7$, hence it follows that k is either 0 or 1.

If $k=1$, then $R_{2k}=5$ and the equation is $x^3+y^3+1+25xy=0$. The solution set is given by

$$\{(27,-19),(-19,27),(5,-1),(-1,5),(0,-1),(-1,0),$$
$$(-1,-5),(-5,-1),(-2,-7),(-7,-2),(-9,-13),(-13,-9)\}$$

and $S_1=[-1,0,5,-5,27,-19,-9,-13]$. The possible arithmetic progressions (with positive difference) are as follows

$$-19,-13,\square,-1,5 \qquad \square,-13,-9,-5,-1, \qquad -13,-9,-5,-1,\square.$$

In the first case we get a 5-term arithmetic progression since $\square=-7$ is an x-coordinate in the solution set. In the last two cases $\square=-17,3$ and these numbers are not x-coordinates from the set, thus we get only 4-term arithmetic progressions.

Integral solutions for fixed z

In what follows we consider the symmetric case of equation S_{a_1,a_2,b_1,b_2}, namely S_{a_1,a_2,a_1,a_2}. We note that the two special equations S_1 and S_2 considered by Kollár and Li belong to this ballpark. Here Runge's method can be applied in an elementary and very explicit way. Let us define two polynomials

$$\begin{aligned} f_1(x,y) &= 9x^2 - 9xy + 9y^2 + (3a_2 - 3z)x + (3a_2 - 3z)y + z^2 + \\ &\quad + a_2 z + z^2 + 9a_1 - 2a_2^2, \\ f_2(x,y) &= 3x + 3y + 2a_2 + z. \end{aligned}$$

Let us rewrite equation S_{a_1,a_2,a_1,a_2} in the form

$$F(x,y) := x^3 + y^3 + 1 - zxy + a_2 x^2 + a_1 x + a_2 y^2 + a_1 y = 0.$$

We can characterize the solutions of equation $F(x,y) = 0$ for fixed values of z.

Integral solutions for fixed z

Theorem 3. *If $(x, y) \in \mathbb{Z}^2$ is a solution of equation $F(x, y) = 0$ for some $z \in \mathbb{Z}$, then we have*

$$x = -\frac{6\,a_2 d_1 - 3\,d_1^2 + 3\,d_1 z \pm \sqrt{H}}{18\,d_1},$$

$$y = -\frac{2}{3}\,a_2 + \frac{1}{3}\,d_1 - x - \frac{1}{3}\,z,$$

where d_1 is an integer dividing $z^3 + 3\,a_2 z^2 + 9\,a_1 z + 18\,a_1 a_2 - 4\,a_2^3 - 27$ and

$$\begin{aligned} H \quad = \quad & -3\,d_1^4 + 12\,d_1 z^3 + 36\,(a_2^2 - 3\,a_1)d_1^2 + 9\,(4\,a_2 d_1 - 3\,d_1^2)z^2 - \\ & -12\,(4\,a_2^3 - 18\,a_1 a_2 + 27)d_1 - 18\,(2\,a_2 d_1^2 - d_1^3 - 6\,a_1 d_1)z. \end{aligned}$$

UNIVERSITY of DEBRECEN

Integral solutions for fixed z

Theorem 3

All integral solutions $(x, y) \in \mathbb{Z}^2$ of the equation $S_1 : xyz = x^3 + y^3 + 1 - x^2 - y^2$ with $-5 \cdot 10^5 \leq z \leq 5 \cdot 10^5$ are given by

z	list of solutions
3565	$[(1171, -2213), (-2213, 1171)]$
2041	$[(671, 229), (229, 671)]$
505	$[(7, -59), (-59, 7)]$
1	$[(1, 1), (1, -1), (-1, 1)]$
-3	$[(-1, -1)]$
-47	$[(-7, -17), (-17, -7)]$
-1007	$[(293, -83), (-83, 293)]$
-2207	$[(-67, -383), (-383, -67)]$
-37407	$[(19061, -8767), (-8767, 19061)]$

.

Integral solutions for fixed z

Theorem 4

All integral solutions $(x, y) \in \mathbb{Z}^2$ of the equation

$$S_2 : xyz = x^3 + y^3 + 1 - 2x^2 - x - 2y^2 - y,$$

with $-5 \cdot 10^5 \leq z \leq 5 \cdot 10^5$ are given by

z	list of solutions
23943	$[(3277, -9073), (-9073, 3277)]$
1095	$[(293, -601), (-601, 293)]$
3	$[(1, -1), (-1, 1)]$
-3	$[(1, 1), (-1, -1)]$
-18183	$[(14183, -8597), (-8597, 14183)]$
-35157	$[(5209, -769), (-769, 5209)]$

UNIVERSITY of DEBRECEN

Parametric solutions

In Theorem 3 there is a cubic expression given by

$$C_{a_1,a_2}(z) := z^3 + 3\,a_2 z^2 + 9\,a_1 z + 18\,a_1 a_2 - 4\,a_2^3 - 27.$$

The idea that we will apply here is based on the simple fact that we need to know some non-trivial factors of $C_{a_1,a_2}(z)$, hence we would like to fix the parameters in such a way that we end up with a reducible expression. In this direction we have that

$$\begin{aligned} C_{a_1,a_1}(z) &= -\big(2\,a_1^2 - a_1 z - z^2 - 6\,a_1 - 3\,z - 9\big)(2\,a_1 + z - 3), \\ C_{a_1,-a_1-2}(z) &= \big(2\,a_1^2 + a_1 z - z^2 + 2\,a_1 + 5\,z + 5\big)(2\,a_1 - z + 1). \end{aligned}$$

We fix z such that the linear terms vanish, that is $z = -2a_1 + 3$ and $z = 2a_1 + 1$, respectively.

Parametric solutions

Substituting these expression into $F(x,y)$ we obtain the following equations

$$\begin{aligned}(a_1x + x^2 + a_1y - xy + y^2 - x - y + 1)(x + y + 1) &= 0,\\ -(a_1x - x^2 + a_1y + xy - y^2 + x + y + 1)(x + y - 1) &= 0.\end{aligned}$$

Therefore if $a_2 = a_1$, then $z = -2a_1 + 3, y = -x - 1$ gives a family of solutions, if $a_2 = -a_1 - 2$, then $z = 2a_1 + 1, y = 1 - x$ provides an infinite family of solutions.

Parametric solutions

Theorem 5

If $a_1 = a_2$, then equation $F(x,y) = 0$ has infinitely many solutions of the form $(x,y,z) = (x, -x-1, -2a_1+3)$. If $a_1 = a_2 = 3m^2+3m+3, z = -2a_1+3$ then the equation has solutions given by

$$(x,y),(y,x) \in \{(-m-1,m),(-m-1,-3m^2-5m-3),$$
$$(m,-3m^2-m-1),(-3m^2-m-1,-6m^2-5m-3),$$
$$(-6m^2-5m-3,-6m^2-7m-4),(-6m^2-7m-4,-3m^2-5m-3)\}.$$

If $a_2 = -a_1 - 2$, then equation $F(x,y) = 0$ has infinitely many solutions of the form $(x,y,z) = (x,-x+1,2a_1+1)$. If $a_2 = -a_1-2, z = 2a_1+1$ and a_1 is of the form $3m^2+3m-1$, then the equation has solutions given by

$$(x,y),(y,x) \in \{(m+1,-m),(m+1,3m^2+5m+1),$$
$$(-m,3m^2+m-1),(3m^2+m-1,6m^2+5m-1),$$
$$(6m^2+5m-1,6m^2+7m),(6m^2+7m,3m^2+5m+1)\}.$$

Parametric solutions + σ_x, σ_y

The second part of Theorem 5 is about the equation

$$xyz = x^3 + y^3 + (-a_1 - 2)x^2 + a_1 x + (-a_1 - 2)y^2 + a_1 y + 1.$$

We know that $(x, 1-x)$ is a solution. Let us apply the method of Kollár and Li.

$$\begin{aligned} (x, 1-x) &\quad \sigma_x \\ (x, -x^2 + (a_1+1)x + 1) &\quad \sigma_y \\ (\text{degree 5 polynomial}, -x^2 + (a_1+1)x + 1) &\quad \sigma_x \\ (\text{degree 5 polynomial}, \text{degree 13 polynomial}) &\quad \sigma_y \\ (\text{degree 34 polynomial}, \text{degree 13 polynomial}). & \end{aligned}$$

Degrees: $u_1 = 1, u_2 = 2, u_n = 3u_{n-1} - u_{n-2}$: Fibonacci numbers having odd indices.

Parametric solutions + σ_x, σ_y

Let us see a concrete example $(x, a_1) = (2, 3)$:

$$\begin{aligned} (2,-1) &\quad \sigma_x \\ (2,5) &\quad \sigma_y \\ (88,5) &\quad \sigma_x \\ (88,140853) &\quad \sigma_y \\ (31754129761534,140853). & \end{aligned}$$

That is we have the solutions $(2, -1)$ and $(31754129761534, 140853)$ in case of the equation

$$xyz = x^3 + y^3 - 5x^2 + 3x - 5y^2 + 3y + 1.$$