Number of solutions to a special type of unit equations in two unknowns II

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purely Exponential Diophantine Equation

$$a^x + b^y = c^z$$

a, b, c fixed positive integers > 1
relatively prime

x, y, z unknown positive integers

- $3^x + 4^y = 5^z$
- $\boxed{\text{unknown} = 1}$ allowed

Basic facts

- $\#\{(x, y, z)\}$ is absolutely finite.

  $\leftarrow$-- Subspace theorem

- $x, y, z < \mathcal{C}_{\text{eff}}(a, b, c)$.

  $\leftarrow$-- p-adic analogue to Baker's method

In recent years, there has been important progress on <u>number of solutions</u>.

Proposition [Bennett,'01] *atmost2pillai*

For any $a, b, c \in \mathbb{N}$; $a, b > 1, \gcd(a, b) = 1,$
there are at most 2 sol.s to

$$a^x - b^y = c \quad x, y \geq 1.$$

- best possible
- special case of Pillai's eq.

Motivation of Part I

3-variable version of *atmost2pillai*

Proposition [M. & Pink,'20] *atmost2*

For any $a, b, c \in \mathbb{N}_{>1}$; $\gcd(a, b, c) = 1$,

there are at most 2 sol.s to

$$a^x + b^y = c^z \quad x, y, z \geq 1,$$

except for $\{a, b\} = \{3, 5\}, c = 2$.

- $3 + 5 = 8 \quad 27 + 5 = 32 \quad 3 + 125 = 128$
- $\exists \infty (a, b, c)$ allowing the eq. to have 2 sol.s

Conjecture [Bennett,'01] *atmost1pillai*

For any $a, b, c \in \mathbb{N}$; $a, b > 1, \mathrm{gcd}(a, b) = 1$,
there is at most 1 sol. to

$$a^x - b^y = c \quad x, y \geq 1,$$

except for the cases:

$2^3 - 3 = 2^5 - 3^3 = 5$ $\quad$ $2^4 - 3 = 2^8 - 3^5 = 13$

$2^3 - 5 = 2^7 - 5^3 = 3$ $\quad$ $3 - 2 = 3^2 - 2^3 = 1$

$13 - 3 = 13^3 - 3^7 = 10$ $\quad$ $91 - 2 = 91^2 - 2^{13} = 89$

The number of exceptional (a, b, c) is proven to be finite by Subspace theorem & abc-conjecture.

Bennett confirmed $atmost1pillai$ (his conjecture) for each of the cases:

- $\cdot\ c \geq b^{2a^2 \log a}$
- $\cdot\ c \leq b^y/6000$ or $c \leq 100$
- $b \equiv \pm 1 \bmod a$ with a prime

  ($\rightsquigarrow$ proving $atmost1pillai$ for a : Fermat primes)

Q Can we prove a 3-variable version of some of the above results?

Conjecture [Scott & Styer,'16] *atmost1*

There is at most 1 sol. to

$$a^x + b^y = c^z,$$

except when (a,b,c) *belongs to*

$$\{\ (5,3,2),(13,3,2),(5,2,3),(7,2,3),(3,2,11),$$
$$(10,3,13),(3,2,35),(89,2,91),(5,2,133),$$
$$(3,2,259),(13,3,2200),(91,2,8283),$$
$$(2^k-1,2,2^k+1); k \geq 2\ (\neq 3)\ \}.$$

- $a,b,c \neq$ perfect powers, $a > b$
- 3-variable version of $atmost1pillai$

Previous works ('56~)

Sierpiński, Jeśmanoiwcz, Dem'janenko, Ko, $\cdots$

<u>R.Scott ('93~)</u>

- seq.s from factorization over $\mathbb{Q}(\sqrt{-a^x b^y})$
- works with R.Styer

★ $c = 2 \Rightarrow atmost 1$

<u>N.Terai, M.Le, P.Yuan, etc. ('94~)</u>

- $^{\exists}r \geq 2$; $a^2 + b^2 = c^r$ *Terai's conj.*
- elementary+Baker+ternary eq. $+ \cdots$

A well-known theorem:

Proposition [Scott,'93]

There is at most 1 sol. to

$$a^x + b^y = 2^z,$$

except for $\{a, b\} = \{3, 5\}, \{13, 3\}$.

- $13 + 3 = 16 \quad 13 + 243 = 256$
- purely algebraic manner in $\mathbb{Q}(\sqrt{-a^x b^y})$

Fundamental result:

Theorem 1

$a \equiv \pm 1 \mod c$ & $b \equiv \pm 1 \mod c$

$\Rightarrow$ *atmost1*

- 3-variable version of one of Bennett's results
- "&" can be replaced by "OR".
- `computation time: 2 weeks`, by

  ASUS computer with a 8-core 11th generation Intel-Core-7 11800H 4.6 GHz processor and with 16 GB of RAM

Corollary 1

$c \in \{2, 3, 6\} \Rightarrow$ *atmost1*

- 『 $p \nmid \mathcal{A} \Rightarrow \mathcal{A} \equiv \pm 1 \mod p$ 』 for $p \in \{2, 3\}$
- **another proof of Scott's result for $c = 2$**

For a set S of primes, we define the S-*part* of a positive integer $\mathcal{A}$ as follows:

$$\mathcal{A}[S] := \prod_{p \in S} p^{\nu_p(\mathcal{A})}.$$

non-explicit but effective generalization:

Theorem 2

Let S be a set of odd prime factors of c.

Assume $a, b \equiv \pm 1 \bmod M_S$ & $c_S > \sqrt{c}$, *where*

(I) $M_S = \prod_{p \in S} p, \quad c_S = \max(\, c[S], c[2] \,)$; *or*

(II) $M_S = 4 \prod_{p \in S} p, \quad c_S = c[S \cup \{2\}]$.

If $a^x + b^y = c^z$ *has* $\mathit{2}$ *sol.s, then* $a, b, c \ll 1$, *or*

$$c_S/\sqrt{c} < \mathcal{C} \quad \& \quad a, b < \exp\Big(\frac{\log \mathcal{C}}{(\log c_S)/\log\sqrt{c} \; -1} \Big),$$

where $\mathcal{C}$ *is some positive absolute const. being effectively computable.*

Restrictions with $\mathcal{C}$ under assuming 2 sol.s

- $S = \{\text{odd primes of } c\}$

$$\overset{\text{(I)or(II)}}{\Rightarrow} \quad M_S \mid c \quad c_S \approx c;$$

$$c/\sqrt{c} \ll 1 \quad a, b \ll 1 \quad \Rightarrow \quad \text{effective Th1}$$

- $S = \{\text{odd primes of } c\} \cap \{3\}$

$$\overset{\text{(I)}}{\Rightarrow} \ M_S \in \{1, 3\} \quad c_S = \max(\, c[2], c[3]\,) > \sqrt{c};$$

$$\max(\, c[2], c[3]\,)\,/\sqrt{c} \ll 1 \quad a, b \ll_c 1$$

$\Rightarrow$ following:

Corollary 2

For any fixed c satisfying

$$\max(\, c[2], c[3]\,) > \sqrt{c},$$

atmost1 is true, except for only finitely many pairs of a and b.

Corollary 3

$$c = p^n \cdot k \;\Rightarrow\; \textit{atmost1}$$

where $p \in \{2, 3\}$, $k \not\equiv 0 \pmod{p}$, $n \geq n_0(k)$.

- $n_0(k)$: const. depending only on k
- $atmost1$ is true for infinitely many values of c.

Main idea with 2 sol.s (c prime)

$$a^x + b^y = c^z \quad a^X + b^Y = c^Z \quad z \le Z$$

$$\rightsquigarrow c^z \mid \mathrm{GCD}(\, a^{e(a)} \pm 1,\, b^{e(b)} \pm 1\,) \cdot \Delta$$

$$e(h) := e_c(h) \text{ least s.t. } h^{e(h)} \equiv \pm 1\,(c)$$

$$\Delta := |x \cdot Y - X \cdot y| \neq 0$$

(This played a central role to prove $atmost2$.)

$$E := \quad e(a) = e(b) \text{ w.l.o.g.}$$

$E = 1 \Rightarrow a, b$: close* to 1 c-adically

Sketch of Proof of Theorem 1

$\nu_c(a^X + b^Y) \ = Z$

$<$ $\cdots$ by c-adic analogue to Baker

$\Rightarrow$ $\boxed{x, y, X, Y \ll_E 1}$ ※ $E \ll_c 1$

$\rightsquigarrow$ $c^z \ll_E (\sqrt{c}^{\,z})^E$

Assume $E = 1$

$\Rightarrow$ $z < 100$ $c < 3 \cdot 10^5$ $(Z < 80000)$

Check $\{a^x + b^y = c^z \,\&\, a^X + b^Y = c^Z$

$\rightsquigarrow$ $(a, b, c) = (5,3,2),(13,3,2),(5,2,3),(7,2,3)$ $\square$

More detail for $c = 2$

Assume $2 \mid c$

$$a \equiv 1 \quad b \equiv -1 \quad (4)$$

$$a^x + b^y = c^z \quad a^X + b^Y = c^Z \quad z \leq Z$$

$$x, y, X, Y \text{ odd}$$

$\Rightarrow \; a^x \equiv -b^y \quad a^X \equiv -b^Y \quad \bmod c^z$

$\rightsquigarrow \; a^{\Delta} \equiv 1 \quad (, b^{\Delta} \equiv 1)$

Since $c^z \mid a^{\Delta} - 1$ & $2 \mid c$,

$$\nu_2(c) \cdot z \leq \nu_2(a^{\Delta} - 1)$$

$$= \nu_2(a-1) + \nu_2(\Delta)$$

$\rightsquigarrow$

$$\nu_2(c) \cdot z \leq \min\Big\{\nu_2(a-1),\, \nu_2(b+1)\Big\} + \nu_2(\Delta)$$

Below, assume $c = 2$.

$$\therefore\ 2^z \;\Big|\; \gcd(a-1, b+1) \cdot \Delta$$

Upper bound for Δ

$$\Delta = |xY - Xy|$$

$$< xY \quad (\text{if } xY > Xy)$$

$$< \frac{\log 2}{\log a} z \cdot \frac{\log 2}{\log b} Z \qquad \because \; a^x < 2^z \quad b^Y < 2^Z$$

$$\ll z \cdot \frac{Z}{\log a \log b}$$

1st application of Baker (sketch)

$$a^X + b^Y = 2^Z$$

$$Z = \nu_2(\, a^X - (-b)^Y \,)$$

$$\ll \frac{\mathrm{LCM}(\, e_c(a), e_c(b) \,)}{\log^2 2} \; \log a \; \log b \;\; \log^2 \mathcal{B}$$

$$\mathcal{B} = \max\{X, Y\}/\mathcal{H}$$

$$\ll \frac{E}{\log^2 2} \; \log a \; \log b \;\; \log^2\!\left(\frac{Z}{\log a \; \log b}\right)$$

$$\therefore \; Z \ll \log a \; \log b$$

$$\Delta \ll z \cdot \frac{Z}{\log a \, \log b} \ll z$$ much smaller than 2^z

$$C := \frac{2^z}{\gcd(2^z, \Delta)} \approx 2^z$$

$$C \mid \gcd(a-1, b+1)$$

2nd application of Baker (sketch)

$$\nu_C(2^Z) = \nu_C(\, a^X - (-b)^Y\,)$$

$$\frac{Z}{z} \ll \frac{\mathsf{LCM}(\, e_C(a), e_C(b)\,)}{\log^2 C} \; \log a \; \log b \;\; \log^2 \mathcal{B}$$

$$\mathcal{B} = \max\{X, Y\}/\mathcal{H} \cdot \log C$$

$$\ll \frac{1}{z^2} \; \log a \; \log b \;\; \log^2\!\left(\frac{Z}{\log a \; \log b} \cdot z\right)$$

$$\therefore \;\; z \cdot Z \ll \log a \; \log b$$

$$z \cdot Z \ll \log(2^z)^{\frac{1}{x}} \cdot \log(2^z)^{\frac{1}{y}} \ll \frac{z^2}{xy}$$

$x, y \ll 1 \quad Z \ll z \quad X, Y \ll 1$

$\Delta \ll 1 \quad C \approx 2^z$

Note that $C \leq a - 1, b + 1$. trivial

$$\therefore\ 2^z \ll (2^z)^{\frac{1}{\max\{x,y\}}}$$

$\underline{x > 1 \text{ or } y > 1}$

$(2/\sqrt{2})^z \ll 1 \quad z \ll 1 \quad Z \ll 1$ ...(1hour)... //

$$\underline{x = 1 \;\; \& \;\; y = 1}$$

$$a + b = 2^z$$

$$a = AC + 1 \quad b = BC - 1 \quad A + B = 2^z/C$$

$$(AC + 1)^X + (BC - 1)^Y = 2^Z$$

$$\boxed{\text{mod } C^2}$$

$$\rightsquigarrow C \mid AX + BY \quad (\text{if } Z \geq 2z)$$

$$\Rightarrow C \ll A + B$$

$$(2/\sqrt{2})^z \ll 1 \quad \ldots(\texttt{50min})\ldots \quad // \quad \Box$$

Another application of Theorem 1:

Theorem 3

c : *Fermat prime** $\Rightarrow$ *atmost1*

- 3 5 17 257 65537
- `computation time: 9 hours`
- Th 1 + $\#\,(\mathbb{Z}/c\mathbb{Z})^{\times}$ power of 2 + c prime
- $2^{2^n}+1 \neq$ prime for $n = 5, \cdots, 32$

Sketch of Proof

$$a^x + b^y = c^z \quad a^X + b^Y = c^Z$$

$$E = e_c(a) = e_c(b) \quad \Delta = |x \cdot Y - X \cdot y|$$

"Th 1 + $\varphi(c)$ power of 2 + c prime"

yields a nice restriction on the parities of x, y, X, Y.

$$E \mid \varphi(c) \quad E \mid \Delta \quad (\because a^{\varphi(c)} \equiv 1,\ a^{\Delta} \equiv \pm 1 \ (c))$$

- $E > 1$ by Th 1
- $2 \mid E$ by $\varphi(c)$ power of 2

$\therefore 2 \mid \Delta$

· $x \not\equiv X \mod 2$ or $y \not\equiv Y \mod 2$

by **Scott's parity result for** $\boxed{c \text{ prime}}$

$\Rightarrow$ x, y : even OR $\underbrace{X, Y : \text{even}}_{\text{assume}}$

$\rightsquigarrow$ $a^{X/2}, b^{Y/2}$: terms of $\pi, Z,$ where $\pi\bar{\pi} = c$;

$E = E(c, Z) \ll \log c$;

$Z \ll z$ by Baker (complex, 2-logs)

Further, $\min\{z, Z\} \ll_c 1$

by π-adic analogue to Baker

(F. Luca's idea for Terai's conj.)

$+$ and more $\rightsquigarrow$ $\{a, b\} = \{c - 2, 2\}$ $\square$

Status of $atmost1$ for each small c

c	**2**	**3**	**5**	6	7	10	**11**	12
status	✓	✓	✓	✓	?	?	?	✓*
13	14	15	**17**	18	19	20	21	22
?	?	?	✓	✓*	?	?	?	?
23	24	26	28	29	30	31	**33**	34
?	✓*	?	?	?	?	?	?	?
35	37	38	39	40	41	42	43	44
?	?	?	?	✓*	?	?	?	?

✓ solved completely

✓* solved except for only finitely many (a, b) being effectively determined

Future work (Part III in progress)

- more investigations to case $E > 1$
- proving $atmost1$ for $c \in \{\text{other values}\}$
- proving $atmost1pillai$ for $a \in \{\text{other values}\}$
- application of abc-conjecture

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Thank you for your attention!