Identities for integer partitions

joint work with Maciej Ulas

Szabolcs Tengely

13 May 2022

UNIVERSITY of DEBRECEN

Table of Contents

Partitions

Let $A \subset \mathbb{N}$ be given and take $n \in \mathbb{N}$. By an A-partition $\lambda = (\lambda_1, \ldots, \lambda_k)$, of a non-negative integer n with parts in A, we mean a representation of n in the form

$$n = \lambda_1 + \ldots + \lambda_k,$$

where $\lambda_i \in A$. The representations of n differing only in order of the terms are counted as one. We also put

$$\mathrm{Part}_A(n) = \{\lambda : \ \lambda \text{ is } A\text{-partition of } n\},$$

and consider the corresponding partition function

$$P_A(n) := \#\,\mathrm{Part}_A(n).$$

It is well know that the ordinary generating function of the sequence $(P_A(n))_{n\in\mathbb{N}}$ takes the form

$$\prod_{a\in A}\frac{1}{1-x^a}=\sum_{n=0}^{\infty}P_A(n)x^n.$$

In particular, if $A=\mathbb{N}$, then $P_A(n)$, simply denoted as $p(n)$, is the famous partition function introduced by Euler and extensively studied by Ramanujan.
If $\mathcal{W}$ is a certain property which can be applied to the parts of a given A-partition λ of a positive integer n, then by $P_A(\mathcal{W},n)$ we denote the number of A-partitions of n which have the property $\mathcal{W}$. Thus, by a partition identity we mean an identity of the form

$$P_{A_1}(\mathcal{W}_1,n)=P_{A_2}(\mathcal{W}_2,n),$$

where $A_1,A_2\subset\mathbb{N}$ and $\mathcal{W}_1,\mathcal{W}_2$ are given properties.

Classical results

We have that

$$\prod_{j=1}^{\infty} \frac{1}{1-x^j} = \sum_{n=0}^{\infty} p(n)x^n.$$

Euler pentagonal number theorem:

$$\prod_{j=1}^{\infty}(1-x^j) = \sum_{-\infty}^{\infty}(-1)^n x^{n(3n+1)/2}.$$

Jacobi triple product identity:

$$\prod_{n=1}^{\infty}(1-x^{2n})(1+x^{2n-1}z^2)(1+x^{2n-1}z^{-2}) = \sum_{-\infty}^{\infty} x^{n^2} z^{2n}.$$

Classical results

Euler's famous identity:

$$P_{\mathbb{N}}(\text{distinct parts}, n) = P_{\mathbb{N}}(\text{odd parts}, n)$$

that is

$$\prod_{n=1}^{\infty}(1+x^n) = \prod_{n=1}^{\infty}\frac{1}{1-x^{2n-1}}.$$

Rogers-Ramanujan type identities:

$$\prod_{n=1}^{\infty}\frac{1}{(1-x^{5n-1})(1-x^{5n-4})} = \sum_{n=1}^{\infty}\frac{x^{n^2}}{(1-x)(1-x^2)\cdot\ldots\cdot(1-x^n)},$$

$$\prod_{n=1}^{\infty}\frac{1}{(1-x^{5n-2})(1-x^{5n-3})} = \sum_{n=1}^{\infty}\frac{x^{n^2+n}}{(1-x)(1-x^2)\cdot\ldots\cdot(1-x^n)}.$$

Polynomial values: $f(m) = P_A(\mathcal{W}, n) \neq 0$

The Ramanujan congruences:

$$p(5n+4) \equiv 0 \pmod 5, \quad p(7n+5) \equiv 0 \pmod 7, \quad p(11n+7) \equiv 0 \pmod{11}.$$

The equations

$$p(n) = 5m, \quad p(n) = 7m, \quad p(n) = 11m$$

have infinitely many solutions in positive integers.
Lovejoy obtained similar results in case of $p(n|$ distinct parts), for example

$$p(26645n + 76| \text{ distinct parts}) \equiv 0 \pmod 5.$$

It was conjectured by Erdős that if m is prime, then there is at least one non-negative integer n_m for which

$$p(n_m) \equiv 0 \pmod{m}.$$

Nicolas, Ruzsa and Sárközy in 1998 constructed a set $A = \{1, 2, 3, 5, 8, 9, 10, 13, \ldots\}$ by recursion such that $P_A(n)$ is even if $n \geq 4$.
In 2000 Ono proved that

$$p\left(\frac{m^k l^3 n + 1}{24}\right) \equiv 0 \pmod{m},$$

where $m \geq 5$ is a prime, k is a positive integer and $\gcd(n, l) = 1$ (for a positive proportion of the primes l).
As a special case one has that

$$p(59^4 \cdot 13n + 111247) \equiv 0 \pmod{13}.$$

Recently (May 2022, arXiv), Zheng considered the function $g(n)$ counting the number of partitions of n in which no part appears exactly once. For example $g(6) = 4$ since

$$6 = 3+3 = 2+2+2 = 2+2+1+1 = 1+1+1+1+1+1.$$

It is proved that

$$g\left(\frac{ml^3n+1}{24}\right) \equiv 0 \pmod{m},$$

where $m \geq 5$ is a prime and $\gcd(n, l) = 1$ (for a positive proportion of the primes l).

As a special case one has

$$g(102487n + 1941) \equiv 0 \pmod{7}.$$

In May 2022 (arXiv) Kronenburg described two algorithms to compute the number of integer partitions of n into exactly m parts.
In May 2022 (arXiv) Binner studied the number of partitions of n into parts not divisible by m. The following formula was obtained

$$P_m(n) = p(n) + \sum_{k\geq 1}(-1)^k \left(p\left(n - \frac{mk(3k-1)}{2}\right) + p\left(n - \frac{mk(3k+1)}{2}\right)\right).$$

If $n = 17$ and $m = 3$, then one has that

$$P_3(17) = p(17) - p(14) - p(11) + p(2).$$

Counting the rationals

Construction by Calkin and Wilf (2000):

$$\{b(n)\}_{n\geq\infty} = \{1, 1, 2, 1, 3, 2, 3, 1, 4, 3, 5, 2, 5, 3, 4, 1, 5, 4, 7, \ldots\}.$$

The fractions $\frac{b(n)}{b(n+1)}$:

$$1, \frac{1}{2}, 2, \frac{1}{3}, \frac{3}{2}, \frac{2}{3}, 3, \frac{1}{4}, \frac{4}{3}, \frac{3}{5}, \frac{5}{2}, \frac{2}{5}, \frac{5}{3}, \frac{3}{4}, 4, \frac{1}{5}, \frac{5}{4}, \frac{4}{7}, \frac{7}{3}, \frac{3}{8}, \frac{8}{5}, \frac{5}{7}, \frac{7}{2}, \frac{2}{7}, \frac{7}{5}, \frac{5}{8}, \frac{8}{3}, \frac{3}{7}, \frac{7}{4}, \frac{4}{5}, 5,$$

$$\frac{1}{6}, \frac{6}{5}, \frac{5}{9}, \frac{9}{4}, \frac{4}{11}, \frac{11}{7}, \frac{7}{10}, \frac{10}{3}, \frac{3}{11}, \frac{11}{8}, \frac{8}{13}, \frac{13}{5}, \frac{5}{12}, \frac{12}{7}, \frac{7}{9}, \frac{9}{2}, \frac{2}{9}, \frac{9}{7}, \frac{7}{12}, \frac{12}{5}, \frac{5}{13}$$

Counting the rationals

The sequence $b(n)$ is the number of ways of writing the integer n as a sum of powers of 2, each power being used at most twice. For example
$5 = 4 + 1 = 2 + 2 + 1$, hence $b(5) = 2$.
Every positive rational occurs once and only once in this list.
One has that $b(0) = 1$ and

$$b(2n+1) = b(n) \qquad b(2n+2) = b(n+1) + b(n).$$

Formulas

If the part are from some "small" sets $\{1, 2, \ldots, m\}$, then there exist formulas (method by Cayley and MacMahon):

$$
\begin{aligned}
P_{\{1\}}(n) &= 1, \\
P_{\{1,2\}}(n) &= \left\lfloor \frac{n}{2} \right\rfloor + 1, \\
P_{\{1,2\}}(n) &= \frac{2n + 3 + (-1)^n}{4}, \\
P_{\{1,2,3\}}(n) &= \frac{(n+3)^2}{12} + \omega(n), \text{ where } \omega(n) \in \{-1/3, -1/12, 0, 1/4\}, \\
P_{\{1,2,3,4\}}(n) &= \left\{ (n+5)(n^2 + n + 22 + 18 \left\lfloor \frac{n}{2} \right\rfloor)/144) \right\}, \\
&\quad \text{where } \{\cdot\} \text{ denotes the nearest integer}
\end{aligned}
$$

The equation $P_3(x) = P_n(y)$ for $n = 4, 5$

Theorem 1 (Ulas-Tengely)
The Diophantine equation $P_3(x) = P_4(y)$ has infinitely many solutions in integers.

Theorem 2 (Ulas-Tengely)
The equation $P_3(x) = P_5(y)$ has only finitely many solutions in positive integers. More precisely, the pair (x, y) is a solution if and only if $(x, y) \in \mathcal{A}$, where

$$\begin{aligned}\mathcal{A} = \{&(1,1),(2,2),(3,3),(5,4),(6,5),(8,6),(16,10),(18,11),(26,14),\\&(45,20),(174,45),(217,51),(457,77),(468,78),(701,97),(10093,388)\}.\end{aligned}$$

Conjecture

There are infinitely many values of $a \in \mathbb{N}_{\geq 4}$ such that for $A = \{1, 2, 3, a\}$, the Diophantine equation $P_3(x) = P_A(y)$ has infinitely many solutions in positive integers.

Proof of Theorem 1

Define $P_{i,6,3}(n) = P_3(6n+i)$ and observe that

$$\begin{aligned} P_{0,6,3}(n) &= 3n^2+3n+1, & P_{1,6,3}(n) &= (n+1)(3n+1),\\ P_{2,6,3}(n) &= (n+1)(3n+2), & P_{3,6,3}(n) &= 3(n+1)^2,\\ P_{4,6,3}(n) &= (n+1)(3n+4), & P_{5,6,3}(n) &= (n+1)(3n+5). \end{aligned}$$

By defining $P_{2i+1,6,4}(n) = P_4(6n+2i+1)$ for $i=0,1,2$ and $P_{2i,12,4}(n) = P_4(12n+2i)$ for $i=0,1,\ldots,5$, we get polynomials as well, e.g.

$$P_{0,12,4}(n) = 12n^3+15n^2+6n+1.$$

Proof of Theorem 1

(i,j)	integral solutions (x,y) of $P_{i,6,3}(x) = P_{2j+1,6,4}(y)$
$(0,0)$	$(0,0)$
$(0,1)$	$(0,0)$
$(1,1)$	$(8,4), (6533,439)$
$(1,2)$	$(293,54)$
$(3,1)$	$((t-1)(2t^2+2t+1), 2(t-1)(t+1)), t \in \mathbb{N}_+$
$(3,2)$	$(2t^3+t-1, 2t^2-1), t \in \mathbb{N}_+$
$(5,1)$	$(5,3)$

(i,j)	integral solutions (x,y) of $P_{i,6,3}(x) = P_{2j,12,4}(y)$
$(0,0)$	$((t-1)(2t^2-t+1), 2(t-1)t), t \in \mathbb{N}_+$
$(1,0)$	$(0,0)$
$(2,1)$	$(0,0)$
$(3,4)$	$(2t^3+3t^2+t-1, t^2+t-1), t \in \mathbb{N}_+$
$(5,2)$	$(0,0)$

Proof of Theorem 2

Special thanks to Nikos Tzanakis sharing ideas to complete the proof of this result.

We have that $P_5(60n+i), i \in \{0, \ldots, 59\}$ is a polynomial in variable n. We need to consider $6 \cdot 60 = 360$ equations of the form $Y^2 =$ quartic. We may apply the MAGMA procedure `IntegralQuarticPoints()` based on a paper by Tzanakis. It worked well in all except the 8 cases, where the MAGMA function failed to determine the complete set of integral solutions. These 8 problematic equations are of the form

$$P_3(6y+i) = P_5(60x+j)$$

for

$$(i,j) \in \mathcal{A} = \{(3,9),(3,12),(3,21),(3,24),(3,33),(3,36),(3,48),(3,57)\}.$$

Proof of Theorem 2

The equations corresponding to $(i,j) = (3,48), (3,57)$ are of the following form

$$\begin{aligned} Y^2 &= u(54000u^3 - 16200u^2 + 1410u - 18), \\ Y^2 &= u(54000u^3 + 16200u^2 + 1410u + 18), \end{aligned}$$

respectively, where $u = x + 1$. In both cases we obtain that u is a square multiplied by a divisor of 18. Therefore we need to handle the equations

$$(2\delta^2 v)^2 = (60\delta u)^3 - 18\delta(60\delta u)^2 + 94\delta^2(60\delta u) - 72\delta^3,$$

where $\delta \in \{\pm 1, \pm 2, \pm 3, \pm 6, \pm 9, \pm 18\}$.

Proof of Theorem 2

In the remaining 6 cases, we observed that the discriminant of $P_3(6y+i) = P_5(60x+j)$ with respect to y is equal to

$$F(u) = 432u^4 + 648u^3 + 282u^2 + 18u,$$

for suitable substitution of the form $u = ax + b$ (depending on values of i, j). Therefore we only need to determine integral points on the curve

$$72u^4 + 108u^3 + 47u^2 + 3u = 30v^2.$$

We obtain that 3 divides u, so $u = 3u_1$ for some integer u_1. We have that

$$u_1(648u_1^3 + 324u_1^2 + 47u_1 + 1) = 30v_1^2, \text{ where } v = 3v_1.$$

The factorization yields the following elliptic curves

$$X^3 + 324\delta X^2 + 30456\delta^2 X + 419904\delta^3 = Y^2, \text{ where } \delta \in \{1, 2, 3, 5, 6, 10, 15, 30\}.$$

Results related to $P_{\{1,2,a\}}(n)$

Earlier results by Ehrhart, Sertöz and Özlük, here we need explicit coefficients.
Let $a \in \mathbb{N}_{\geq 3}$ and put $A = \{1, 2, a\}$. If $a = 2c$ for some $c \in \mathbb{N}_{\geq 2}$ then

$$P_A(4cn+i) = 2cn^2 + \left(c + 2\left\lfloor \frac{i}{2} \right\rfloor + 2\right) n + \begin{cases} \left\lfloor \frac{i+2}{2} \right\rfloor, & i \in \{0, \dots, 2c-1\} \\ 2\left\lfloor \frac{i}{2} \right\rfloor + 2 - a, & i \in \{2c, \dots, 4c-1\} \end{cases}.$$

If $a = 2c + 1$ for some $c \in \mathbb{N}_+$ then

$$\begin{aligned} P_A(2(2c+1)n + i) &= (2c+1)n^2 + (c+i+2)n \\ &+ \begin{cases} \left\lfloor \frac{i+2}{2} \right\rfloor, & i \in \{0, \dots, 2c\} \\ i + 1 - a, & i \in \{2c+1, \dots, 4c+1\}. \end{cases} \end{aligned}$$

Results related to $P_{\{1,2,a\}}(n)$

Theorem 3 (Ulas-Tengely)
Let $a \in \mathbb{N}_{\geq 3}$ and put $A = \{1, 2, a\}$. The Diophantine equation $y^2 = P_A(x)$ has infinitely many solutions in positive integers.

The proof is based on Pell-equations. Let a be even, i.e., $a = 2c$ for some c. We have that $P_A(4cn) = 2cn^2 + (c+2)n + 1 = y^2$. The lines through $(0, 1)$ can be written as $y = mn + 1$. Therefore we get that $2cn^2 + (c+2)n + 1 = (mn+1)^2$, that is $n = 0$ or $n = \frac{c+2-2m}{m^2-2c}$. Here $m = u/v$ is a rational parameter, so we have that

$$n = \frac{(c+2)v^2 - 2uv}{u^2 - 2cv^2}.$$

For our assumption, $a = 2c$ is not a square, then we consider the Pell-equation $u^2 - 2cv^2 = 1$ and denote the sequence of positive integer solutions by (u_k, v_k). In this case it follows that $n = (c+2)v_k^2 - 2u_kv_k$ and $y = (c+2)u_kv_k - 2u_k^2$.

Results related to $P_{\{1,2,a\}}(n)$

Theorem 4 (Ulas-Tengely)
Let $a, b \in \mathbb{N}_{\geq 3}, a < b$ such that a, b are divisible by 4 and either $a/2$ or $b/2$ is not a square. Put $A = \{1, 2, a\}, B = \{1, 2, b\}$. The Diophantine equation $P_A(x) = P_B(y)$ has infinitely many solutions in positive integers.

Let $a = 2s$ and $b = 2t$. We have that

$$\begin{aligned} P_A(4sn) &= 2sn^2 + (s+2)n + 1, \\ P_B(4tm) &= 2tm^2 + (t+2)m + 1. \end{aligned}$$

Suppose that $a/2 = s$ is not a square. We get that

$$m = \frac{(s+2)/2uv - (t+2)/2v^2}{v^2 - su^2}.$$

The integer s is not a square, hence we consider the sequence of positive solutions (u_k, v_k) of the Pell-equation $v^2 - su^2 = 1$. For these solutions we have $m = (s+2)/2u_kv_k - (t+2)/2v_k^2$ and $n = (s+2)/2u_k^2 - (t+2)/2u_kv_k$.

Conjecture

Let $a, b \in \mathbb{N}_{\geq 3}, a < b$ and put $A = \{1, 2, a\}, B = \{1, 2, b\}$. The Diophantine equation $P_A(x) = P_B(y)$ has infinitely many solutions in positive integers.

The equation $P_A(x) = P_B(y)$

Theorem 5 (Ulas-Tengely)
Let $a \in \mathbb{N}_{\geq 3}, b \in \mathbb{N}_{\geq 4}$ and put $A = \{1, 2, a\}, B = \{1, 2, 3, 4, b\}$. If $a \equiv 1, 2, 5, 7, 11, 10 \pmod{12}$ and $b = 4a$, then the Diophantine equation $P_A(x) = P_B(y)$ has infinitely many solutions in positive integers.

We present the values of i, j such that the polynomial $P_A(2am + i) - P_B(3bn + j)$ is reducible.
Let $a = 6k + 1, b = 4a, i = 11k, j = 3(8k - 1)$. Then
$P_A(2am + i) - P_B(3bn + j) = R_1(m, n)R_2(m, n)$, where

$$R_1(m, n) = m - 3(6k + 5)n^2 - 2(9k + 7)n - 4k + 1,$$
$$R_2(m, n) = (6k + 1)m + 3(6k + 1)^2 n^2 + 2(6k + 1)(9k + 1)n + 24k^2 + 12k + 1.$$

Thus, if $m = 3(6k + 5)n^2 + 2(9k + 7)n + 4k - 1$ then $P_A(2am + i) = P_B(3bn + j)$ and our equation has infinitely many solutions.

The equation $P_A(x) = P_B(y)$

We proved that for many choices of sequences A, B, the corresponding Diophantine equation $P_A(x) = P_B(y)$ has infinitely many solutions in positive integers. However, in each case under consideration we had $\min\{\#A, \#B\} \le 3$.

Question

Let $A, B \subset \mathbb{N}_+$. Let us suppose that the Diophantine equation $P_A(x) = P_B(y)$ has infinitely many (non-trivial) solutions in positive integers. How large the number $\min\{\#A, \#B\}$ can be?

UNIVERSITY of DEBRECEN

The equation $P_A(x) = P_B(y)$

Let us explain what a trivial solution means. More precisely, if for example $A = \{1, pa_2, \ldots, pa_k\}$ then if $P_A(pn)$ is a non-zero, then in each representation

$$1 \cdot x_1 + \sum_{i=2}^{k} pa_i x_i = pn$$

we need to have $p|x_1$ and thus we get a representation

$$1 \cdot y_1 + \sum_{i=2}^{k} a_i x_i = n.$$

It is clear that this mapping can be reversed. Thus, by taking $B = \{1, a_2, \ldots, a_k\}$ we have the boring identity $P_A(pn) = P_B(n)$.

The equation $P_A(x) = P_B(y)$

We considered equations of the form $P_A(x) = P_B(y)$, where A, B are sets having 5 elements from $\{1, 2, \ldots, 10\}$ and one of the elements is 1. We searched for reducible polynomials $P_A(x) - P_B(y)$ having a linear or quadratic factor. We implemented a parallel algorithm and used SageMath on a machine having 16 cores. It took about 10 hours to determine the appropriate polynomials. There are 44982 such cases. Among these polynomials we looked for examples providing infinitely many integral solutions. To reduce the time of computation a timeout was set to be 60 seconds. There are 392 cases for which the 60 seconds were not sufficient to compute the result. There are 2338 quadratic equations that yield infinitely many integral solutions and 2100 linear equations that provide parametric solutions. However, even in the case of reducibility we sometimes get factors without positive integer solutions.

The equation $P_A(x) = P_B(y)$

Let $A = \{1, 2, 4, 5, 6\}$ and $B = \{1, 4, 6, 9, 10\}$. Here we obtain that $P_A(60m + 22) - P_B(180n + 111)$ is, up to a constant factor, equal to $f_1(m, n)f_2(m, n)$, where

$$
\begin{aligned}
f_1(m, n) &= 150m^2 + 450n^2 + 155m + 630n + 259, \\
f_2(m, n) &= 30m^2 - 90n^2 + 31m - 126n - 36.
\end{aligned}
$$

The equation $f_1(m, n) = 0$ has no solution modulo 5. The equation $f_2(m, n) = 0$ has infinitely many integral solutions. However, all are negative.

The equation $P_A(x) = P_B(y)$

Let $A = \{1, 2, 3, 4, 6\}$, $B = \{1, 2, 4, 5, 10\}$. It follows that $P_A(12m+1) - P_B(20n+1) = 1/6 h_1(m,n) h_2(m,n)$, where

$$\begin{aligned} h_1(m,n) &= 6m^2 + 10n^2 + 9m + 12n + 5 \\ h_2(m,n) &= 6m^2 - 10n^2 + 9m - 12n. \end{aligned}$$

The equation $h_1(m,n) = 0$ can be written as

$$15(36m+27)^2 + (180n+108)^2 = 6399,$$

and it follows that the only integral solution is given by $(m,n) = (-1,-1)$. The equation $h_2(m,n) = 0$ has infinitely many positive integral solutions, the two smallest being $(m,n) = (2928, 2268), (11252256, 8715960)$.

The equation $y^2 = P_A(x)$

In case of $\#A = 5$ there is a large number of sets such that $P_A(L_A n + i)$ is a square of a polynomial in n. More precisely, with the constraint $\max(A) \leq 15$, there are exactly 119 different pairs (A, i) such that $P_A(L_A n + i)$ is a square of a polynomial with integer coefficients. For example, if $A = \{1, 2, 8, 10, 15\}$, then $L_A = 120$ and for $i = 1, 11, 41, 43, 73, 83, 91, 113$ we have $P_A(L_A n + i)$ is a square of a polynomial. In particular,

$$P_A(120n + 1) = (4n + 1)^2(15n + 1)^2.$$

The equation $y^2 = P_A(x)$

A	L_A	i
$\{1,2,8,10,15\}$	120	$1,11,41,43,73,83,91,113$
$\{1,4,5,10,12\}$	60	$12,16,36,52$
$\{1,4,8,9,12\}$	72	$1,13,19,25,37,43,49,61,67$
$\{1,5,6,8,10\}$	120	$2,8,13,17,32,37,53,58,73,77,82,88,97,98,112,113$
$\{2,3,7,8,14\}$	168	$32,102,144,158$
$\{2,4,5,6,10\}$	60	$12,16,17,21,36,41,52,57$
$\{3,4,6,9,12\}$	36	$3,7,11,27,31,35$
$\{3,5,6,9,15\}$	90	$18,23,24,28,29,34,54,59,64,78,83,88$
$\{4,5,6,12,15\}$	60	$27,51$
$\{4,7,9,12,14\}$	252	$58,64,142,148,226,232$
$\{5,6,8,9,10\}$	360	$8,29,53,74,89,98,104,113,128,149,173,194,209,$
		$218,224,233,248,269,293,314,329,338,344,353$
$\{5,7,9,14,15\}$	630	$47,113,173,197,257,323,383,407,467,533,593,617$
$\{7,8,10,14,15\}$	840	$182,212,364,422,574,604,812,814$

UNIVERSITY of DEBRECEN

The equation $y^2 = P_A(x)$

We were able to find only one set A with 7 elements, $\max(A) \leq 10$ such that $y^2 = P_A(x)$ has infinitely many solutions in positive integers. More precisely, if $A = \{1, 2, 4, 5, 8, 9, 10\}$ then

$$P_A(360n + 95) = 25(3n + 1)^2(18n + 5)^2(36n + 13)(40n + 13),$$
$$P_A(360n + 226) = 25(3n + 2)^2(18n + 13)^2(36n + 23)(40n + 27).$$

One can easily check that the factor $(36n + 13)(40n + 13)$ is a square infinitely often. The smallest values of n which makes this factor a square, are $n = 0, 494, 712842, \ldots$.

The equation $y^2 = P_k(x)$

A difficult and still unsolved question is whether the number $p(n)$ can be a perfect power. Let us recall that $p(n)$ counts the number of all partitions of n, i.e.,

$$\prod_{n=1}^{\infty} \frac{1}{1-x^n} = \sum_{n=0}^{\infty} p(n)x^n.$$

In other words, we do not know any example of $n \geq 2$ such that $y^k = p(n)$ for some $k \in \mathbb{N}_{\geq 2}$. In fact Zhi-Wei Sun conjectured that the equation $y^k = p(n)$ has no solutions in positive integers n, y, k with $k \geq 2$. Let us also note that Alekseyev checked that there are no solutions with $n \leq 10^8$.

The equation $y^2 = P_k(x)$

Theorem 6 (Ulas-Tengely)
The equation $y^2 = P_5(x)$ has only finitely many solutions in positive integers. More precisely, the pair (x, y) is a solution if and only if $(x, y) = (1, 1), (2027, 77129)$.

We have 60 curves of the form $y^2 = P_5(60n + i), i \in \{0, \ldots, 59\}$. If $i \in \{5, 20, 25, 40\}$ the corresponding quartic has no $\mathbb{Q}_5$-rational points, and thus has no rational points at all. The solution $(1, 1)$ comes from the equation $y^2 = P_5(60n + 1)$ with $n = 0$. The solution $(2027, 77129)$ comes from the solution $(n, y) = (33, 77129)$ of the equation $y^2 = P_5(60n + 47)$.

The equation $y^2 = P_k(x)$

The procedure `IntegralQuarticPoints()` works well, except in 6 special cases. Here we get warnings about time-consuming final enumerations. The 6 problematic polynomials correspond to $i \in \{21, 24, 48, 51, 54, 57\}$.
The equations (up to multiplication by 16) corresponding to $i = 21, 24$ give the following equations

$$\begin{aligned} 5y^2 &= u(36u^3 + 108u^2 + 34u + 12), \quad u = 5(2x+1), \\ 5y^2 &= u(36u^3 - 108u^2 + 34u - 12), \quad u = 2(5x+3), \end{aligned}$$

respectively. Hence we need to resolve the following elliptic equations

$$Y^2 = X^3 + 108\delta X^2 + 3384\delta^2 X + 15552\delta^3,$$

where δ divides 60.

The equation $y^2 = P_k(x)$

For $i \in \{48, 51, 54, 57\}$ after the substitution $u = 2(x+1)$ we get the following quartic equations

$$\begin{aligned}
y^2 &= u(4500u^3 - 2700u^2 + 470u - 12),\\
y^2 &= u(4500u^3 - 900u^2 - 70u + 4),\\
y^2 &= u(4500u^3 + 900u^2 - 70u - 4),\\
y^2 &= u(4500u^3 + 2700u^2 + 470u + 12).
\end{aligned}$$

UNIVERSITY of DEBRECEN

The equation $y^2 = P_k(x)$

The case of the equation $y^2 = P_6(x)$ is far more difficult. To get the solutions we need to consider 60 genus 2 curves

$$C_i: \; y^2 = P_6(60x + i), \quad i = 0, \ldots, 59.$$

Let $J_i = \mathrm{Jac}(C_i)$ be the Jacobian of the curve C_i and by r_i denote the rank of J_i. We checked that $r_i \leq 5$ for $0 \leq i \leq 59$.

The equation $y^2 = P_k(x)$

r	values of i such that $r_i \leq r$
0	$3, 14, 34, 47, 50, 51, 55, 59$
1	$18, 22, 27, 32, 35, 38, 41, 43, 44, 45, 46, 54$
2	$0, 7, 8, 9, 15, 23, 24, 25, 26, 28, 29, 30, 33, 36, 37, 39, 40, 42, 49, 52, 53, 57, 58$
3	$2, 5, 6, 11, 17, 31, 48$
4	$4, 10, 13, 16, 19, 20, 21, 56$
5	$1, 12$

Upper bounds for the $\mathbb{Q}$-rank of the Jacobian J_i of the curve
$C_i: \ y^2 = P_6(60x + i)$ for $i = 0, \ldots, 59$.

UNIVERSITY of DEBRECEN

The equation $y^2 = P_k(x)$

It is curious that the polynomial $P_6(60x + i)$ is reducible (in the ring $\mathbb{Q}[x]$) for $i \in \{40, \ldots, 59\}$ and thus, instead of working with genus two curve we need to play with certain curves of the type $y^2 = Q_i(x)$, where Q_i is quartic.
Among the above curves there are some for which we were not able to obtain Mordell-Weil bases, these are as follows

$$i \in \{15, 16, 23, 24, 27, 28, 29, 31, 32, 33, 35, 36, 38, 39\}.$$

The most interesting one may be the hyperelliptic curve given by

$$y^2 = 12x^5 + 1125x^4 + 41960x^3 + 778050x^2 + 7171020x + 26276400,$$

corresponding to the curve $C_{27}: \quad y^2 = P_6(60x + 27)$. In this case the rank is 1, however we were unable to find a generator of the Mordell-Weil group.