On the difference of k-generalized Lucas numbers and powers of 2

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GENERALIZATION

Fibonacci, and Lucas numbers

$$F_0 = 0,\ F_1 = 1, \qquad F_n = F_{n-1} + F_{n-2},$$
$$L_0 = 2,\ L_1 = 1, \qquad L_n = L_{n-1} + L_{n-2},$$

$$\Downarrow\ k \geq 2\ \Downarrow$$

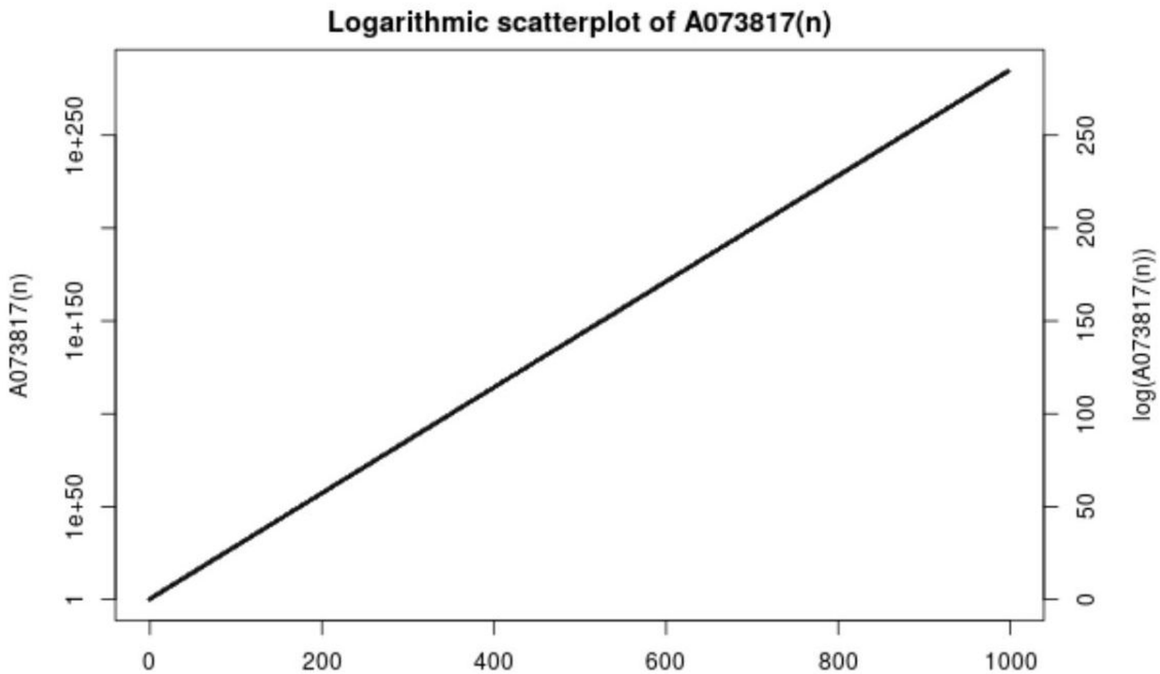
$$F_n^{(k)} = F_{n-1}^{(k)} + F_{n-2}^{(k)} + \cdots + F_{n-k}^{(k)},$$
$$L_n^{(k)} = L_{n-1}^{(k)} + L_{n-2}^{(k)} + \cdots + L_{n-k}^{(k)},$$

$$F_0^{(k)} = 0, F_1^{(k)} = 1, \quad \text{and} \quad F_n^{(k)} = 2^{n-2} \quad \text{if} \quad 2 \leq n \leq k-1,$$

$$L_0^{(k)} = k, \quad \text{and} \quad L_n^{(k)} = 2^n - 1 \quad \text{for} \quad 1 \leq n \leq k-1.$$

Example: $k = 4$

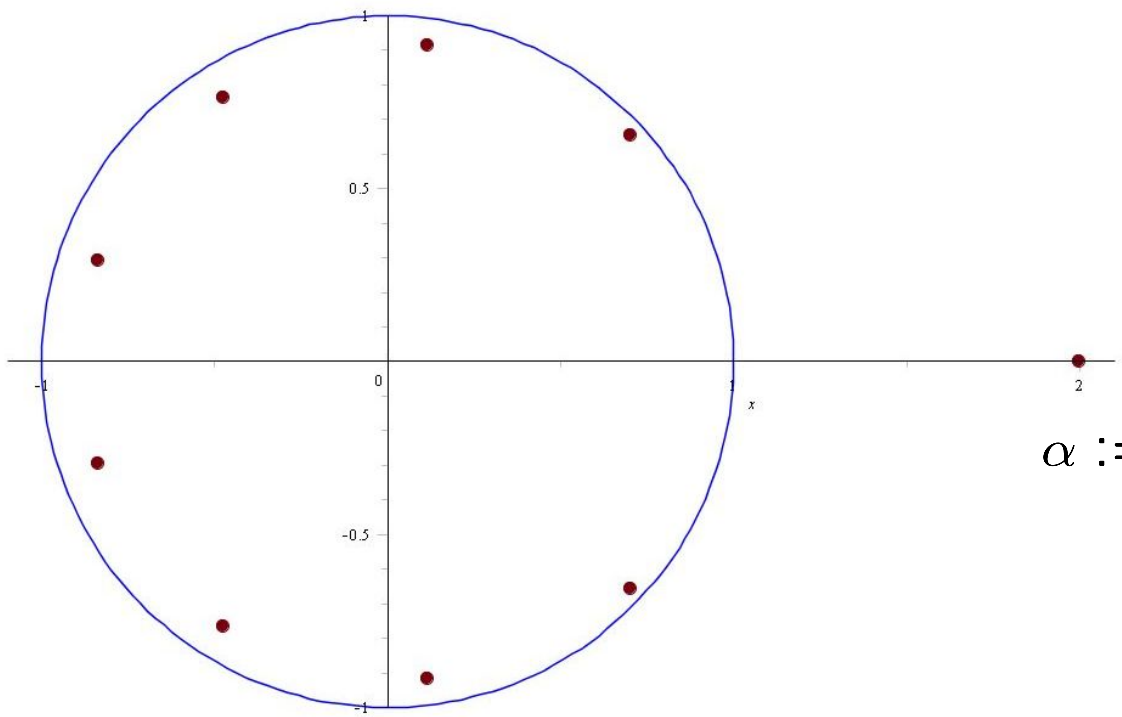
n	0	1	2	3	4	5	6	7	8	...	OEIS
$F_n^{(4)}$	**0**	**1**	**1**	**2**	4	8	15	29	56	...	A000078
$L_n^{(4)}$	**4**	**1**	**3**	**7**	15	26	51	99	191	...	A073817



Characteristic polynomial:

$$p_k(x) = x^k - x^{k-1} - \cdots - x - 1 = \prod_{j=1}^{k} (x - \alpha_j)$$

$k = 9$:



$\alpha := \alpha_1$

Explicit (Binet) formula ($\longleftarrow$ FTHLR)

$$F_n^{(k)} = \sum_{j=1}^{k} \frac{g_k(\alpha_j)}{\alpha_j} \alpha_j^n \quad \text{for all} \quad n \geq 0,$$

where

$$g_k(x) = \frac{x-1}{2+(k+1)(x-2)}$$

(Dresden and Du*).

For k-generalized Lucas numbers

$$\mathbf{L_n^{(k)}} = \alpha_\mathbf{1}^\mathbf{n} + \alpha_\mathbf{2}^\mathbf{n} + \cdots + \alpha_\mathbf{k}^\mathbf{n}. \tag{1}$$

*A simplified Binet formula for k-generalized Fibonacci numbers, (2014)

THE PROBLEM

Find terms **CLOSE** to 2^m (in k-GLN)

Example: equality $F_n^{(k)} = 2^m$ (in k-GFN)

$F_1^{(k)} = 1$, $F_2^{(k)} = 1$, $F_3^{(k)} = 2$, $\ldots$, $F_k^{(k)} = 2^{k-2}$, and $F_{k+1}^{(k)} = 2^{k-1}$.
$\longleftarrow$ *trivial solutions*

THEOREM (Bravo-Luca[†])

The only non-trivial solution to $F_n^{(k)} = 2^m$ is $F_6 = 8$.

[†] Powers of two in generalized Fibonacci sequences, (2012)

Example: equality $L_n^{(k)} = 2^m$ (in k-GLN)

L_n : $\boxed{2}$,$\boxed{1}$,3,$\boxed{4}$,7,11,18,...

$L_n^{(3)}$: 3,$\boxed{1}$,3,7,11,21,...

$L_n^{(4)}$: $\boxed{4}$,$\boxed{1}$,3,7,15,26,...

$L_0^{(2^m)} = 2^m$, and $L_1^{(k)} = 1 \longleftarrow$ *trivial solutions*

COROLLARY (Açikel-Irmak-Sz[‡])

No non-trivial solution apart from $L_3 = 4$.

[‡]The k-generalized Lucas numbers close to a power of 2 (202?)

First we considered $L_n^{(k)} = 2^m + t$ with $|t| \leq t_u = 9$, but ...

Definition 1. *(Chern, Cui[§]) An integer n is said to be close to a positive integer m if n satisfies*

$$|n - m| < \sqrt{m}. \qquad (!)$$

Chern, Cui: 8 solutions to $|F_n - 2^m| < 2^{m/2}$, the largest one: $|34 - 2^5| < 2^{5/2}$.

Bravo, Gomez, Herrera[¶]: extended for $F_n^{(k)}$

Question: **What terms of the k-generalized Lucas sequence are close to a power of 2?**

[§]Fibonacci numbers close to a power of 2, (2014)

[¶]k-Fibonacci numbers close to a power of 2, (2021)

THE RESULT

Main theorem (Açikel-Irmak-Sz.)

The non-trivial solutions (i.e $n \geq k+1$) of the inequality

$$\left| L_n^{(k)} - 2^m \right| < 2^{m/2} \qquad (2)$$

in positive integers n, k, m with $k \geq 2$ are given in the table

k	2	2	2	2	2	2	3	3	3
n	3	4	6	7	10	13	7	8	9
m	2	3	4	5	7	9	6	7	8

,

or $k \geq 5$, $n = m$, and $k+1 \leq n < k+t_0$, where $t = t_0 \in \mathbb{R}^+$ satisfies the equation

$$2^{(k+t)/2} = (k+t) \cdot 2^{t-1} + 1.$$

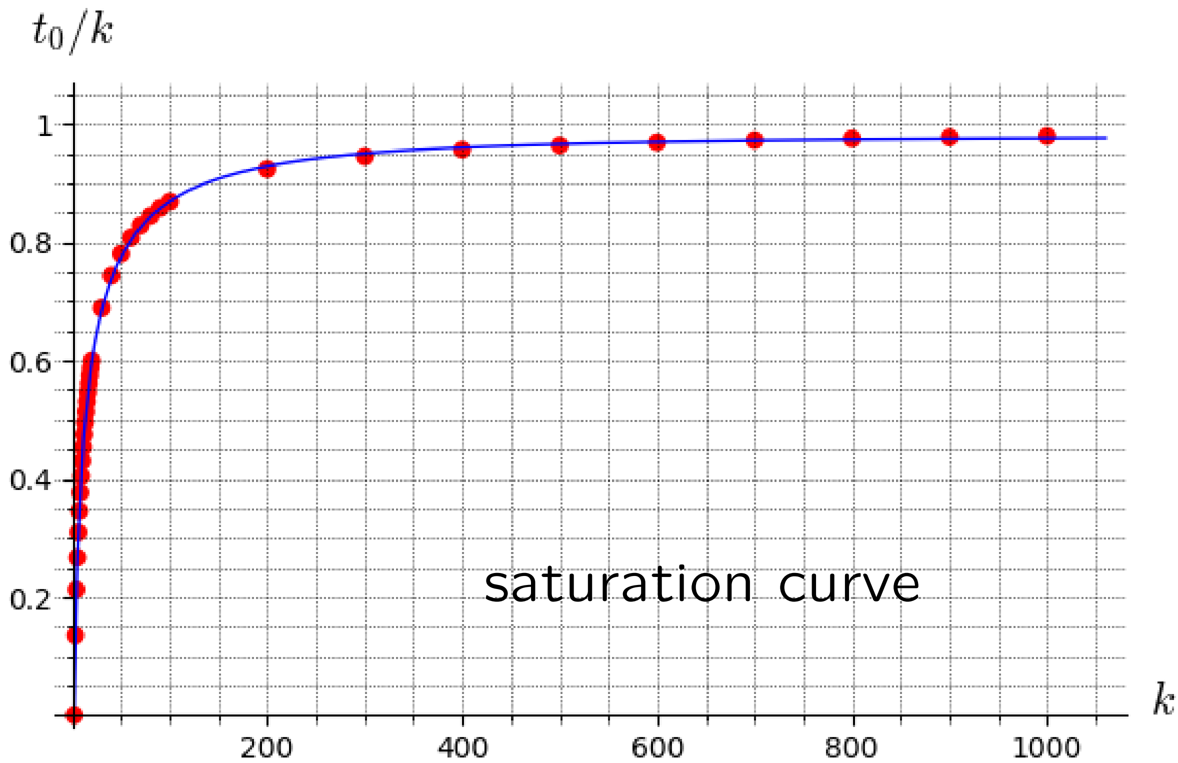
Remarks:

- Equivalent form: $L_n^{(k)} = 2^m + t \quad \text{with} \quad |t| < 2^{m/2}$.
- $L_0^{(k)} = k$, where $k = 2, 3, \ldots$, it is sometimes close to some 2^m (sometimes not)
- $L_k^{(k)} = 2^k - 1$ (first not initial value) $\Longrightarrow L_n^{(k)} = 2^n - 1$ ($n = 1, 2 \ldots, k$) are close to 2^n
- **trivial solutions** (k, n, m) to

$$\left| L_n^{(k)} - 2^m \right| < 2^{m/2}$$

are characterized by $2 \leq k$, $0 \leq n \leq k$.

+ remark: **case $\mathbf{n} = \mathbf{m}$**, when $k \geq 5$, $k+1 \leq n < k+t_0$ $t = t_0 \in \mathbb{R}^+$ satisfies $2^{(k+t)/2} = (k+t) \cdot 2^{t-1} + 1$.



$$\frac{t_0}{k} \approx 2.131 \left(1 - e^{-(k/3.962)^{0.336}}\right) - 1.151, \qquad R = 0.9998$$

PRELIMINARIES

To the proof of the main theorem

- Linear forms in logarithms (Matveev)
- LLL algorithm
- Lemmata (later) + this theorem (Acikel, Irmak, Sz.)

Let $r = \lfloor n/(k+1) \rfloor$. Then we have

$$L_n^{(k)} = 2^n - 1 - n \sum_{j=1}^{r} \frac{(-1)^{j-1}}{j} \binom{n-kj-1}{j-1} 2^{n-(k+1)j}.$$

$1 \le n \le k$: $r = 0$, and $L_n^{(k)} = 2^k - 1$,

$k+1 \le n = k+t \le 2k+1$: $r = 1 \Rightarrow L_n^{(k)} = 2^{k+t} - 1 - (k+t)2^{t-1}$

DETOUR

Analoguous formula for k-generalized Fibonacci numbers

$$L_n^{(k)} = 2^n - 1 + \sum_{j=1}^{r} (-1)^j \left[\binom{n-kj}{j} + k\binom{n-kj-1}{j-1} \right] 2^{n-(k+1)j}.$$

Theorem (Cooper, Howard‖)

Let $r_1 = \lfloor (n-1)/(k+1) \rfloor$. Then

$$F_n^{(k)} = 2^{n-2} + \sum_{j=1}^{r_1} (-1)^j \left[\binom{n-jk}{j} - \binom{n-jk-2}{j-2} \right] 2^{n-(k+1)j-2}.$$

‖Some identities for r-Fibonacci numbers, (2011) RETURN

to lemmata with k-generalized Lucas numbers

- Lemma 0: $L_{n+1}^{(k)} = 2L_n^{(k)} - L_{n-k}^{(k)} \quad \text{for} \quad n \geq k$

- Lemma 1: If $n \geq k$, then $L_{n+1}^{(k)} < 2L_n^{(k)}$

- Lemma 2: $L_n^{(k)} < 2^n$, $n \geq 1$

- Lemma 3: $\alpha^{n-1} \leq L_n^{(k)} < \alpha^{n+1}$, $n \geq 1$

- Lemma 4: $\left| L_n^{(k)} - \alpha^n \right| < k - 1$

SKETCH

of the sketch of the proof

$n \geq k+1$ (non-trivial solutions)

$$\left| L_n^{(k)} - 2^m \right| < 2^{m/2}$$

$$\Downarrow \text{ Baker method}$$

$$m \leq n < ub(k) = c_1 \cdot k^{c_2}$$

$$\swarrow \qquad\qquad\qquad \searrow$$

$$m < n \qquad\qquad\qquad m = n$$

$$\swarrow \qquad \searrow$$

$$k \geq 129 \qquad 2 \leq k \leq 128$$

(Lemma 2: $\longrightarrow L_n^{(k)} \xrightarrow{\quad < \quad} 2^n \longrightarrow$, consequently $m \leq n$)

BEGINNING

Sketch of the proof, assume $\left|L_n^{(k)} - 2^m\right| < 2^{m/2}$

$n \geq k+1$, $\alpha = \alpha_1$: dominating zero

$$0 < |2^m - \alpha^n| \leq |2^m - L_n^{(k)}| + |L_n^{(k)} - \alpha^n| < 2^{m/2} + (k-1).$$

$$0 < \left|\frac{2^m}{\alpha^n} - 1\right| < \frac{k-1}{\alpha^n} + \frac{2^{m/2}}{\alpha^n} < \frac{n-1}{\alpha^n} + \frac{2^{m/2}}{\alpha^n} < \frac{3}{\alpha^{n/2}}$$

$\Downarrow$ Matveev + calc.

$$\mathbf{m} \leq \mathbf{n} < \underbrace{4.7 \cdot 10^{11} \cdot k^3 (\log k)^2}_{ub(k)}$$

$n > m$

k is LARGE ($k \geq 129$), $\left| L_n^{(k)} - 2^m \right| < 2^{m/2}$

$$m < n < 4.7 \cdot 10^{11} \cdot k^3 (\log k)^2 < \mathbf{2^{k/2}}$$

$$\begin{aligned} |\, 2^n - 2^m \,| &= |\, (2^n - \alpha^n) + (\alpha^n - L_n^{(k)}) + (L_n^{(k)} - 2^m) \,| \\ &< \frac{2^{n+1}}{2^{k/2}} + k - 1 + 2^{m/2} \end{aligned}$$

$$|\, 1 - 2^{m-n} \,| < \frac{2}{2^{k/2}} + \frac{k-1}{2^n} + \frac{2^{m/2}}{2^n} < \frac{3}{2^{k/2}},$$

hence

$$\frac{3}{2^{k/2}} > |\, 1 - 2^{m-n} \,| \geq \frac{1}{2} \qquad (\Rightarrow\Leftarrow)$$

$n > m$

k is small $(2 \leq k \leq 128)$, $\left|L_n^{(k)} - 2^m\right| < 2^{m/2}$

Recall that $n < 4.7 \cdot 10^{11} \cdot k^3 (\log k)^2$, and

$$0 < \left|\frac{2^m}{\alpha^n} - 1\right| = \left|e^{m \log 2 - n \log \alpha} - 1\right| < \frac{3}{\alpha^{n/2}} \quad < \frac{3}{4}$$

Put $z = m \log 2 - n \log \alpha$, $(\alpha = \alpha(k))$

$$|z| = |m \log 2 - n \log \alpha| < \frac{6}{\alpha^{n/2}}.$$

LLL $\Downarrow$ for each eligible k

Summary: if $k = 2$, then $n \leq 45$,
if $3 \leq k \leq 128$, then $n \leq 2k + 30$.

$+$ Verification: $\left|L_n^{(k)} - 2^m\right| < 2^{m/2} \Rightarrow$ solutions in the table

$n = m$

$$\text{Plan,}\quad \left|L_n^{(k)} - 2^n\right| < 2^{n/2}$$

This is true for $n = 1, 2, \ldots, k$. ($L_n^{(k)} = 2^n - 1$)

We show that if the subscript n is increased, then

- for some $1 \le t \le k$ the term $L_{k+t}^{(k)}$ leaves the closeness of 2^{k+t}, ($n = k + t$)
- and then never returns for larger subscripts.

Put $D_n = 2^n - 2^{n/2}$. We investigate (first with $n = k + t$)

$$D_n < L_n^{(k)} < 2^n$$

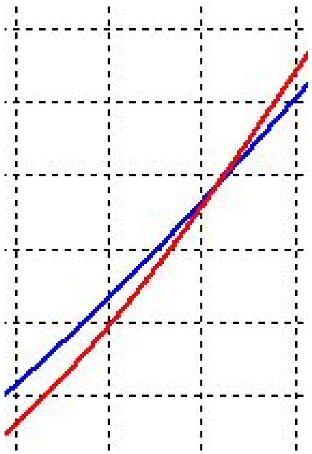
$$
\begin{array}{rcl}
2^n - 2^{n/2} = D_n & < & L_n^{(k)} \\
& \Downarrow & n = k + t,\ 1 \le t \le k \\
2^{k+t} - 2^{(k+t)/2} & < & 2^{k+t} - (k+t)2^{t-1} - 1 \\
\underbrace{(k+t)2^{t-1} + 1}_{g(t)} & < & \underbrace{2^{(k+t)/2}}_{f(t)}
\end{array}
$$

$\mathbf{f(t)}$, $\mathbf{g(t)}$: continuous funct. in $t \in \mathbb{R}$, increasing, convex funct.

$g(0) = k/2 + 1 < 2^{k/2} = f(0)$

$g(k) = k \cdot 2^k + 1 > 2^k = f(k)$

$\Rightarrow \exists! t_0 \in \mathbb{R} : \ g(t_0) = f(t_0)$

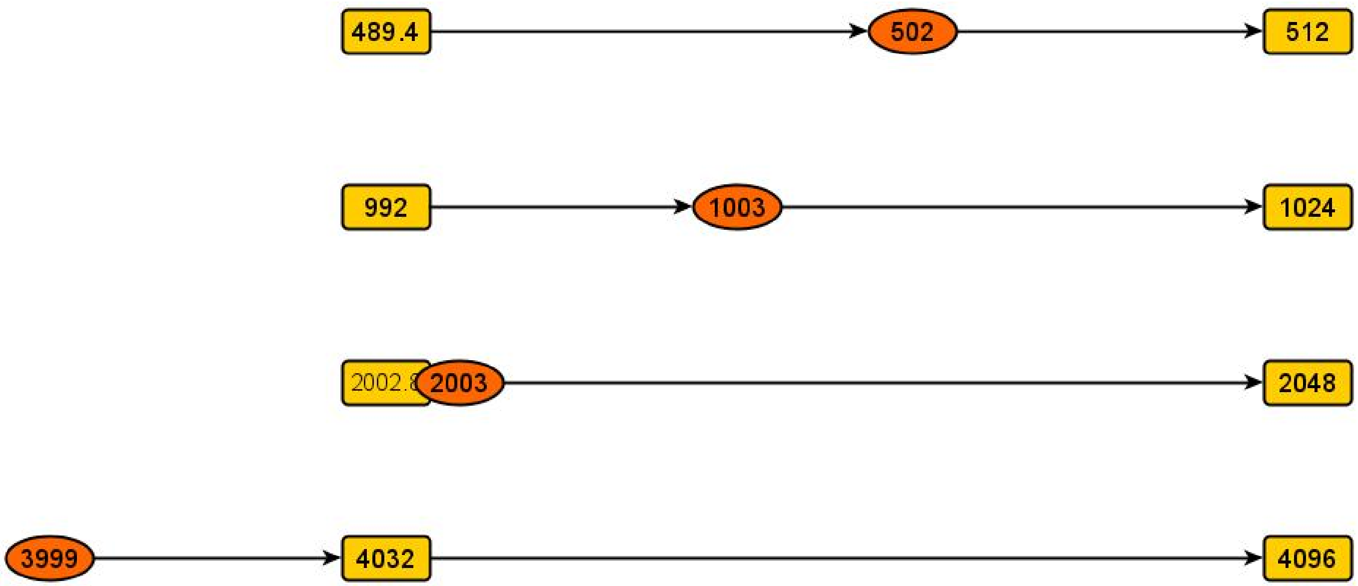


$n = m$

$$t \leq k, \qquad D_{k+t} < L_{k+t}^{(k)} = 2^{k+t} - (k+t)2^{t-1} - 1 < 2^{k+t}$$

k = 8

$n = k + t$	8	9	10	11	12
t	0	1	2	3	4
$L_n^{(k)}$	255	502	1003	2003	3999



$n = m$

Complication: $L_{k+t}^{(k)} = 2^{k+t} - (k+t)2^{t-1} - 1$ is not valid if $t \geq k+2$

Assume that $L_n^{(k)} < D_n = 2^n - 2^{n/2}$, ($\longrightarrow$ left the closeness of...)

Then

$$L_{n+1}^{(k)} < 2L_n^{(k)} < 2D_n < D_{n+1}.$$

(LHS): $L_{n+1}^{(k)} = 2L_n^{(k)} - L_{n-k}^{(k)} < 2L_n^{(k)}$,

(RHS): $2D_n = 2^{n+1} - 2^{n/2+1} < 2^{n+1} - 2^{(n+1)/2} = D_{n+1}$.

READY!

OUTLOOK

More general definition for closeness

Definition 2. *(Açikel, Irmak, Sz.)*

Let $0 < \varepsilon < 1$ *be a real number. We say that a real number* ν *is* ε-**close to** *a real number* μ *if*

$$|\nu - \mu| < |\mu|^{\varepsilon}.$$

$\left|L_n^{(k)} - 2^m\right| < 2^{\varepsilon m}$: most things works (more and less), for example

$$0 < \left|\frac{2^m}{\alpha^n} - 1\right| < \frac{c_1 + 2^{\delta\varepsilon}\alpha^{\varepsilon}}{\alpha^{(1-\varepsilon)n}},$$

but. . .

If $\varepsilon \approx 1$, then case $n = m$ is more problematic, we need to apply

$$L_n^{(k)} = 2^n - 1 - n \sum_{j=1}^{r} \frac{(-1)^{j-1}}{j} \binom{n-kj-1}{j-1} 2^{n-(k+1)j}.$$

with $r \geq 2$ (or use other approach).

For example, if $2k + 2 \leq n \leq 3k + 2$, then

$$L_n^{(k)} = 2^n - 1 - n\left(2^{n-k-1} - (n-2k-1)2^{n-2k-3}\right),$$

so

$$2^n - 2^{n/2} < 2^n - 1 - n\left(2^{n-k-1} - (n-2k-1)2^{n-2k-3}\right)$$

?


Thank You!