Online Number Theory Seminar

15 December 2023. – 17:00-17:50

G. Soydan:

On the solutions of a class of generalized Fermat equations of signature $(2, 2n, 3)$

Fix nonzero integers A, B and C. For given positive integers p, q, r satisfying $1/p+1/q+1/r<1$, the generalized Fermat equation

$$Ax^p + By^q = Cz^r \tag{1}$$

has only finitely many primitive integer solutions. Modern techniques coming from Galois representations and modular forms (methods of Frey–Helle-gouarch curves and variants of Ribet's level-lowering theorem, and of course, the modularity of elliptic curves or abelian varieties over the rationals or totally real number fields) allow to give partial (sometimes complete) results concerning the set of solutions to (1). Recently, two survey papers concerning equation (1) when $ABC = 1$ have been published by Bennett et al.

In this talk, after giving a short survey about generalized Fermat equations, we present some results on the solutions of the Diophantine equations

$$ax^2 + y^{2n} = 4z^3, \quad x, y, z \in \mathbb{Z},\ \gcd(x, y) = 1,\ n \in \mathbb{N}_{\geq 2},$$

and

$$x^2 + ay^{2n} = 4z^3, \quad x, y, z \in \mathbb{Z},\ \gcd(x, y) = 1,\ n \in \mathbb{N}_{\geq 2},$$

where the class number of $\mathbb{Q}(\sqrt{-a})$ with $a \in \{7, 11, 19, 43, 67, 163\}$ is 1.

Our motivation was to extend the former results (and methods) of Bruin, Chen and Dahmen, by considering (1) with (A, B, C)'s different from $(1, 1, 1)$ (assuming for simplicity that the class number of $\mathbb{Q}(\sqrt{-AB})$ is one). This is based on joint work with Karolina Chałupka and Andrzej Dąbrowski.