On the solutions of a class of generalized Fermat equations of signature $(2, 2n, 3)$

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(This is a joint work with K. Chałupka and A. Dąbrowski)

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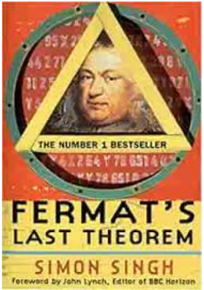
1.Introduction and motivation

A mathematical adventure that started with Fermat in 1637 and ended with Andrew Wiles in 1995:

Theorem 1 (Fermat's last theorem)

The equation $x^p + y^p = z^p$ has no solutions in non-zero integers x, y, z for $p \geq 3$.



- A generalization of Fermat's last theorem:

Conjecture 1 (Beal conjecture)

The equation $x^p + y^q = z^r$ has no solutions in non-zero mutually coprime integers x, y, z for $p, q, r \geq 3$.

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- Andrew Beal is a Dallas banker who has a general interest in mathematics.
- Beal has personally funded a standing prize of $ 1 million USD for its proof or disproof.

- For given positive integers p, q, r satisfying $1/p + 1/q + 1/r < 1$, the generalized Fermat equation

$$Ax^p + By^q = Cz^r \tag{1}$$

has only finitely many primitive integer solutions [Darmon & Granville, 1997].

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  has only finitely many primitive integer solutions [Darmon & Granville, 1997].
- $A = B = C = 1$ and $(p, q, r) = (n, n, n)$: Fermat's equation
- $A = B = C = 1$ and $y = 1$: Catalan's equation

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the case $1/p + 1/q + 1/r = 1$

$(p, q, r) \in \{(2, 6, 3), (2, 4, 4), (3, 3, 3), (4, 4, 2), (2, 3, 6)\}$: Each case corresponds to an elliptic curve of rank 0.

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the case $(p, q, r) = (3, 3, 3)$ and $(A, B, C) = (1, 1, 1)$

Now we consider the equation $x^3 + y^3 = z^3$. The transformation

$$x = \frac{6}{X} + \frac{Y}{6X}, \; y = \frac{6}{X} - \frac{Y}{6X}$$

yields the elliptic curve

$$Y^2 = X^3 - 432.$$

All rational solutions of the above curve are $(X, Y) = (12, 36), (12, 36)$ and $\mathcal{O}$. But none of them does not give any solution to the original equation.

1. Introduction and Motivation

the case $1/p + 1/q + 1/r > 1$

$(p, q, r) \in \{(2, 2, r), (2, q, 2), (2, 3, 3), (2, 3, 4), (2, 4, 3), (2, 3, 5)\}$: No solution or infinitely many solutions.

the case $1/p + 1/q + 1/r > 1$

$(p, q, r) \in \{(2, 2, r), (2, q, 2), (2, 3, 3), (2, 3, 4), (2, 4, 3), (2, 3, 5)\}$: No solution or infinitely many solutions.

the case $(A, B, C) = (1, 1, 1)$ and $(p, q, r) = (2, 2, 2)$

This case corresponds to the equation $x^2 + y^2 = z^2$, which has infinitely many solutions.

five small solutions (for the case $1/p + 1/q + 1/r < 1$)

$1^n + 2^3 = 3^2$
$2^5 + 7^2 = 3^4$
$7^3 + 13^2 = 2^9$
$2^7 + 17^3 = 71^2$
$3^5 + 11^4 = 122^2$
(Kelly, Scott and de Weger all found these examples independently.)

1. Introduction and Motivation

five large solutions (for the case $1/p + 1/q + 1/r < 1$)

$17^7 + 76271^3 = 21063928^2$
$1414^3 + 2213459^2 = 65^7$
$9262^3 + 15312283^2 = 113^7$
$43^8 + 96222^3 = 30042907^2$
$33^8 + 1549034^2 = 15613^3$
(Beukers and Zagier have found these examples.)

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- His proof is based on deep results about Galois representations associated to elliptic curves and modular forms.
- The method of using such results to deal with Diophantine problems, is called the modular approach. After Wiles' proof, the original strategy was strengthened and many mathematicians achieved great success in solving other equations that previously seemed hard.
- As a result of these efforts, the generalized Fermat equation

$$Ax^p + By^q = Cz^r, \text{ with } 1/p + 1/q + 1/r < 1, \tag{2}$$

where $p, q, r \in \mathbb{Z}_{\geq 2}$, A, B, C are non-zero integers and x, y, z are unknown integers became a new area of interest.

- Modern techniques coming from Galois representations and modular forms:
    1. Methods of Frey–Hellegouarch curves and variants of Ribet's level-lowering theorem.
    2. The modularity of elliptic curves or abelian varieties over the rationals or totally real number fields.

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  1. Methods of Frey–Hellegouarch curves and variants of Ribet's level-lowering theorem.
  2. The modularity of elliptic curves or abelian varieties over the rationals or totally real number fields.
- Modern techniques allow to give partial (sometimes complete) results concerning the set of solutions to generalized Fermat equation (usually, when a radical of ABC is small),
- at least when (p, q, r) is of the type (n, n, n), $(n, n, 2)$, $(n, n, 3)$, $(2n, 2n, 5)$, $(2, 4, n)$, $(2, 6, n)$, $(2, n, 4)$, $(2, n, 6)$, $(3, 3, p)$, $(2, 2n, 3)$, $(2, 2n, 5)$.

- Here, note that the notation $\{p, q, r\}$ implies that all permutations of the ordered triple $\{p, q, r\}$ are taken into account.

Some known results with $(A, B, C) = (1, 1, 1)$

$\{n, n, n\}$ and $n \geq 3$: Wiles and Taylor (Fermat's last theorem) (1995).
$\{n, n, 2\}$: Darmon and Merel (for n prime ≥ 7) (1997), Poonen (for $n = 5, 6, 9$) (1998).
$\{n, n, 3\}$: Darmon and Merel (for n prime ≥ 7) (1997), Lucas (19th century) (for $n = 4$) and Poonen (for $n = 5$) (1998).
$\{3, 3, n\}$: Kraus (for $17 \leq n \leq 10000$) (1993), Bruin (for $n = 4, 5$) (2000,2003), Chen and Siksek (for $17 \leq n \leq 10^9$) (2009), Dahmen (for $n = 7, 11, 13$) (2008).
$(2, n, 4)$: Application of Bennett-Skinner (2004), includes $(4, n, 4)$ by Darmon (1993).

Some known results: continued

$(2,4,n)$: Ellenberg (for prime $n \geq 211$) (2004) and Ghioca (for $n = 7$) (see also Poonen Schaefer, Stoll-2007).
$(2n,2n,5)$: Bennett (for $n \geq 7$ and $n = 2$) (2005), Bruin for $n = 3$ (2000) and $n = 5$ follows from Fermat's last theorem.
$(2,2n,3)$: Chen (for n prime and $7 < n < 1000$ and $n \neq 31$) (2008), Dahmen (the case $n = 31$ and $n \equiv 5 \pmod 6$) (2011)
$(2,2n,5)$: Chen (for $n > 17$ prime and $n \equiv 1 \pmod 4$) (2010).
$\{2,4,6\}$: Bruin (1999).

Some known results: continued

$\{2,4,5\}$: Bruin, $2^5+7^2=3^4$, $3^5+11^4=122^2$ (2003).
$\{2,3,9\}$: Bruin, $13^2+7^3=2^9$ (2005).
$\{2,3,8\}$: Bruin, $1^8+2^3=3^2$, $43^8+96222^3=30042907^2$,
$33^8+1549034^2=15613^3$ (1999,2003).
$\{2,3,7\}$: Poonen, Schaefer and Stoll, $1^7+2^3=3^2$, $2^7+17^3=71^2$,
$17^7+76271^3=21063928^2$, $9262^3+15312283^2=113^7$ (2007).
$\{2,6,n\}$: Bennett and Chen (2012), Bennett et al. (2015) (for the case $n\geq 3$)

Some known results: continued

$(3j, 3k, n)$, $j, k \geq 2$, $n \geq 3$: Immediate from Kraus (1998)
$(3, 3, 2n)$, $n \geq 2$: Bennett et al. (2015)
$(3, 6, n)$, $n \geq 2$: Bennett et al.(2015)
$(2, 2n, k)$, $k \in \{9, 10, 15\}$, $n \geq 2$: Bennett et al. (2015)
$(4, 2n, 3)$, $n \geq 2$: Bennett et al. (2015)
$(2m, 2n, 3)$: Bennett et al. (2015) ($n \equiv 3 \pmod 4$, $m \geq 2$)
$(2, 4n, 3)$: Bennett et al.(2015) ($n \equiv \pm 2 \pmod 5$ *or* $n \equiv \pm 2, \pm 4 \pmod{13}$).
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- Survey papers about solving the generalized Fermat equation when $ABC = 1$: [Bennett, Chen, Dahmen, Yazdani-2015], [Bennett, Mihăilescu, Siksek- 2016].

- When $1/p + 1/q + 1/r$ is close to one, for example, consider the equations

$$x^2 + y^3 = z^5, \; x^2 + y^3 = z^7, \; x^2 + y^3 = z^8,$$

then one needs new methods (Chabauty method or its refinements [Bruin-1999,2003], or a combination of Chabauty type method with a modular approach [Poonen, Schaefer, Stoll-2007], [Freitas, Naskręcki, Stoll-2020]).

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- In 2022, we studied the Diophantine equation

$$ax^2 + y^{2n} = 4z^3, \quad x, y, z \in \mathbb{Z}, \; \gcd(x, y) = 1, \; n \in \mathbb{N}_{\geq 2}, \qquad (3)$$

where the class number of $\mathbb{Q}(\sqrt{-a})$ with $a \in \{7, 11, 19, 43, 67, 163\}$ is 1, [Chałupka, Dąbrowski, Soydan-JNT-2022].

- In this talk, we first consider the Diophantine equation

$$x^2 + ay^{2n} = 4z^3, \quad x, y, z \in \mathbb{Z},\ \gcd(x, y) = 1,\ n \in \mathbb{N}_{\geq 2}, \tag{4}$$

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- Why do we work on these equations?

- $x^2 + y^{2n} = z^3$ (Bennett, Bruin, Chen, Dahmen, Yazdani, 1999-2015). It is known that this equation has no solutions for a family of n's of natural density one.

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- Our motivation:
  1. To extend the above results (and methods) of Bruin, Chen and Dahmen, by considering some Diophantine equations $Ax^2 + By^{2n} = Cz^3$ with (A, B, C)'s different from $(1, 1, 1)$ (assuming for simplicity that the class number of $\mathbb{Q}(\sqrt{-AB})$ is one).

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  2. To extend our previous results about the Diophantine equation

$$ax^2 + b^{2n} = 4y^k,\ k > 3 \text{ tek asal},\ x, y \in \mathbb{Z},\ n, k \in \mathbb{N},\ (x, y) = 1,$$

[Dąbrowski, Günhan, Soydan-JNT-2020].

1. Introduction and motivation

- In the above work, we suppose that $a \in \{7, 11, 19, 43, 67, 163\}$ and b is an odd prime. In the new work, we fix $k = 3$, but b is arbitrary.

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- Why were we unable to handle the Diophantine equations $7x^2 + y^{2n+1} = 4z^3$ and $x^2 + 7y^{2n+1} = 4z^3$?

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  1. In 2007, Poonen, Schaefer and Stoll find the primitive integer solutions to $x^2 + y^7 = z^3$. Their method combine the modular method together with determination of rational points on certain genus-3 algebraic curves. This case (and possible generalizations to $Ax^2 + By^7 = Cz^3$) is very difficult.

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  2. In 2020, Freitas, Naskrecki and Stoll considered a general Diophantine equation $x^2 + y^p = z^3$ (with p any prime > 7). They follow and refine the arguments of Poonen, Schaefer and Stoll by combining new ideas around the modular method with recent approaches to determination of the set of rational points on certain algebraic curves.

As a result, they were able to find (under GRH) the complete set of solutions of the Diophantine equation $x^2 + y^p = z^3$ only for $p = 11$.

very recent progresses

- In 2000, Darmon described a program to study the generalized Fermat equation using modularity of abelian varieties of GL_2-type over totally real fields. The original approach was based on hard open conjectures, which have made it difficult to apply in practice. This is called Darmon's program.

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- On 10 th of August 2023 and 14 th of August 2023, Billerey, Chen, Dieulefait and Freitas put two papers on arxiv.org.
- In their first paper, building on the progress surrounding the modular method from the last two decades, they analyze and expand the current limits of this program by developing all the necessary ingredients to use Frey abelian varieties for new Diophantine applications.

very recent progresses

- In particular, they deal with all but the fifth and last step in the modular method for Fermat equations of signature (r, r, p) in almost full generality.

very recent progresses

- In particular, they deal with all but the fifth and last step in the modular method for Fermat equations of signature (r, r, p) in almost full generality.
- As an application, for all integers $n \geq 2$, they give a resolution of the generalized Fermat equation

$$x^{11} + y^{11} = z^n$$

for solutions (a, b, c) such that $a + b$ satisfies certain 2- or 11-adic conditions.

very recent progresses

- And also the tools developed can be viewed as an advance in addressing a difficulty not treated in Darmon's original program: even assuming 'big image' conjectures about residual Galois representations, one still needs to find a method to eliminate Hilbert newforms at the Serre level which do not have complex multiplication.

very recent progresses

- And also the tools developed can be viewed as an advance in addressing a difficulty not treated in Darmon's original program: even assuming 'big image' conjectures about residual Galois representations, one still needs to find a method to eliminate Hilbert newforms at the Serre level which do not have complex multiplication.
- In fact, they are able to reduce the problem of solving $x^5 + y^5 = z^p$ to Darmon's 'big image conjecture', thus completing a line of ideas suggested in his original program, and notably only needing the Cartan case of his conjecture.

very recent progresses

- In their second paper, as a first application, they use a multi-Frey approach combining two Frey elliptic curves over totally real fields, a Frey hyperelliptic curve over $\mathbb{Q}$ due to Kraus, and ideas from the Darmon program to give a complete resolution of the generalized Fermat equation

$$x^7 + y^7 = 3z^n$$

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- In their second paper, as a first application, they use a multi-Frey approach combining two Frey elliptic curves over totally real fields, a Frey hyperelliptic curve over $\mathbb{Q}$ due to Kraus, and ideas from the Darmon program to give a complete resolution of the generalized Fermat equation
$$x^7 + y^7 = 3z^n$$
for all integers $n \geq 2$.
- Moreover, they explain how the use of higher dimensional Frey abelian varieties allows a more efficient proof of this result due to additional structures that they afford, compared to using only Frey elliptic curves.

very recent progresses

- As a second application, they use some of these additional structures that Frey abelian varieties possess to show that a full resolution of the generalized Fermat equation

$$x^7 + y^7 = z^n$$

depends only on the Cartan case of Darmon's big image conjecture. In the process, they solve the previous equation for solutions (a, b, c) such that a + b satisfies certain 2- or 7-adic conditions and all $n \geq 2$.

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- In July 2023, a survey paper about Darmon's program was published by A. Koutsianas & I. Chen. For the details, please see this survey paper.

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Theorem 2 (Chałupka, Dąbrowski, Soydan-?)

The Diophantine equation

$$x^2 + ay^{2n} = 4z^3, \quad x, y, z \in \mathbb{Z},\ \gcd(x, y) = 1,\ n \in \mathbb{N}_{\geq 2},$$

has no solutions where the class number of $\mathbb{Q}(\sqrt{-a})$ *with* $a \in \{11, 19, 43, 67, 163\}$ *is 1.*

Theorem 3 (Chałupka, Dąbrowski, Soydan-?)

Let x, y, z be coprime integers such that $x^2+7y^4=4z^3$. Then there are rational numbers s, t such that one of the following holds.

$$\begin{aligned}
x &= \pm(-s^4-8ts^3+18t^2s^2+24t^3s-9t^4)\\
&\quad(-405t^8-108st^7-504s^2t^6+252s^3t^5-294s^4t^4-84s^5t^3\\
&\quad-56s^6t^2+4s^7t-5s^8),\\
y &= \pm(s^2+3t^2)(-s^4+6ts^3+18t^2s^2-18t^3s-9t^4),\\
z &= (162t^8-108st^7+252s^2t^6+252s^3t^5+84s^4t^4-84s^5t^3\\
&\quad+28s^6t^2+4s^7t+2s^8),
\end{aligned} \tag{5}$$

$$
\begin{aligned}
x &= \pm(1/32)(s^4+21t^4)(441t^8-714s^4t^4+s^8) \\
y &= (3/4)st(s^4-21t^4), \\
z &= (1/16)(441t^8+294s^4t^4+s^8),
\end{aligned}
\tag{6}
$$

$$
\begin{aligned}
x &= \pm(1/32)(3s^4+7t^4)(9s^8-714t^4s^4+49t^8) \\
y &= (3/4)st(3s^4-7t^4), \\
z &= (1/16)(9s^8+294t^4s^4+49t^8).
\end{aligned}
\tag{7}
$$

Theorem 4 (Chałupka, Dąbrowski, Soydan-?)

Any solution to the Diophantine equation $x^2 + 7y^6 = 4z^3$ in coprime integers x, y, z is of the type

$$(x_m, y_m, z_m) = (\pm\omega_m(P)/4d_m^3, \pm\psi_m(P)/d_m, \pm\varphi_m(P)/4d_m^2)$$

for some positive integer m, where $P = (8, 20)$, φ_m, ψ_m and ω_m denote the division polynomials associated to the elliptic curve $Y^2 = X^3 - 112$, and $d_m := \gcd(\pm\omega_m(P)/4, \pm\psi_m(P), \pm\varphi_m(P)/4)$.

Theorem 5 (Chałupka, Dąbrowski, Soydan-?)

The Diophantine equation $x^2 + 7y^8 = 4z^3$ *has the following non-trivial solutions* (x, y, z)*:* $(\pm 5, \pm 1, 2)$*,* $(\pm 16690170427, \pm 105, 4114726)$ *and* $(\pm 165997441137915, \pm 481, 1902746962)$.

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Theorem 6 (Chałupka, Dąbrowski, Soydan-?)

Assume the abc conjecture. Then for a positive proportion of primes p*, all non-trivial solutions to the Diophantine equation* $x^2 + 7y^{2p} = 4z^3$ *in coprime integers* x,y,z *are given by* $(x, y, z) = (\pm 5, \pm 1, 2)$.

Theorem 7 (Chałupka, Dąbrowski, Soydan-?)

Let n be any integer ≥ 2. The Diophantine equation $x^2 + 7y^{2n} = 4z^{12}$ has no solutions in coprime integers x, y, z.

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3. The sketch for the proof (Theorem 2)

- By Theorem 2, we see that the Diophantine equation $x^2 + ay^{2n} = 4z^3$ has no solutions where the class number of $\mathbb{Q}(\sqrt{-a})$ with $a \in \{11, 19, 43, 67, 163\}$ is 1.
- As the class number of $\mathbb{Q}(\sqrt{-a})$ with $a \in \{7, 11, 19, 43, 67, 163\}$ is 1, we have the following factorization for the left side of the eq. $x^2 + ay^{2n} = 4z^3$

$$\frac{x + y^n\sqrt{-a}}{2} \cdot \frac{x - y^n\sqrt{-a}}{2} = z^3.$$

- Now we have

$$\frac{x + y^n\sqrt{-a}}{2} = \left(\frac{u + v\sqrt{-a}}{2}\right)^3,$$

where u, v are odd rational integers. Note that $\gcd(u, v) = 1$. Equating the real and imaginer parts, we obtain the following result.

Lemma 1 (Chałupka, Dąbrowski, Soydan-2022)

Suppose that (x, y, z) is a solution to $x^2 + ay^{2n} = 4z^3$. Then

$$(x, y^n, z) = \left(\frac{u(u^2 - 3av^2)}{4}, \frac{v(3u^2 - av^2)}{4}, \frac{u^2 + av^2}{4}\right) \tag{8}$$

for some odd $u, v \in \mathbb{Z}$ with $\gcd(u, v) = 1$.

3. The sketch for the proof (Theorem 2)

- By Lemma 1, we have $u(u^2 - 3av^2) = 4y^n$ or $v(3u^2 - av^2) = 4y^n$. Now, if $a \in \{11, 19, 43, 67, 163\}$, then $u(u^2 - 3av^2)$ is congruent to 0 modulo 8, while $4y^n$ is congruent to 4 modulo 8, a contradiction. So, the proof is completed.

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- Hence, this lemma completes the proof of Theorem 2.
- So, we only need to consider the Diophantine equation

$$x^2+7y^{2n}=4z^3.$$

- Here we consider the Diophantine equation $x^2 + 7y^4 = 4z^3$.

3. The sketch for the proof (Theorem 3)

- Here we consider the Diophantine equation $x^2 + 7y^4 = 4z^3$.
- We obtain all families of solutions to the title equation (variants of Zagier's result in the case $x^2 + y^4 = z^3$).

- Here we consider the Diophantine equation $x^2 + 7y^6 = 4z^3$.

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- The above equation corresponds to the elliptic curve

$$E: \; Y^2 = X^3 - 2^4 7.$$

  By MAGMA, we obtain that the Mordell-Weil group of E is cyclic infinite, generated by the point $P = (8, 20)$.
- Each solution (x, y, z) of the Diophantine equation $x^2 + 7y^6 = 4z^3$ in coprime integers x, y, z leads to a rational point $(4z/y^2, 4x/y^3)$ on the elliptic curve E. We can obtain all such solutions (up to the signs of x and y) considering integer multiplicities mP of the point P. Now using some known results about elliptic curves, we are done.

3. The sketch for the proof (Theorem 5)

- Here we consider the Diophantine equation

$$7x^2 + y^8 = 4z^3.$$

Any primitive solution of the Diophantine equation $7x^2 + y^8 = 4z^3$ satisfies, of course, the equation $7x^2 + (y^2)^4 = 4z^3$. Hence using Theorem 3, we obtain formulas describing x, y^2 and z.

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- In particular we have the following formulas for y^2:

$$\begin{aligned} y^2 &= \pm(s^2+3t^2)(-s^4+6ts^3+18t^2s^2-18t^3s-9t^4),\\ y^2 &= (3/4)st(s^4-21t^4),\\ y^2 &= (3/4)st(3s^4-7t^4). \end{aligned}$$

Note that $t = 0$ implies $y = 0$. Therefore, nontrivial solutions correspond to affine rational points on one of the following genus two curves:

- Therefore, nontrivial solutions correspond to affine rational points on one of the following genus two curves:

$$\mathcal{C}_5 : Y^2 = (X^2+3)(-X^4+6X^3+18X^2-18X-9),$$
$$\mathcal{C}_6 : Y^2 = -(X^2+3)(-X^4+6X^3+18X^2-18X-9),$$
$$\mathcal{C}_7 : Y^2 = X^5 - 3^5 7X,$$
$$\mathcal{C}_8 : Y^2 = X^5 - 3^7 7X.$$

Definition 2

Let V be a variety defined over $\mathbb{Q}$. V is everywhere locally solvable (ELS) if the set $V(\mathbb{Q}_p)$ is nonempty for all places $p \leq \infty$ of $\mathbb{Q}$.

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Let V be a variety defined over $\mathbb{Q}$. V is everywhere locally solvable (ELS) if the set $V(\mathbb{Q}_p)$ is nonempty for all places $p \leq \infty$ of $\mathbb{Q}$.

- ELS is necessary for existence of $\mathbb{Q}$-points, but sufficient!
- We check that the curve $\mathcal{C}_5$ has no rational points. Indeed, the MAGMA commmand `HasPointsEverywhereLocally`$(f, 2)$ gives $\mathcal{C}_5(\mathbb{Q}_3) = \emptyset$.

3. The sketch for the proof (Theorem 5)

- In 1941, Claude Chabauty proved the finiteness of the number of rational points on curves of genus $g > 0$ with a jacobian of Mordell-Weil rank $< g$ over $\mathbb{Q}$.

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- This is a method for finding the rational points on a curve C of genus at least 2, that applies when the Mordell-Weil group of $Jac(C)$ has rank less than the genus of C. It involves doing local calculations at some prime where C has good reduction.

3. The sketch for the proof (Theorem 5)

- We use Magma subroutines `Chabauty` and `Chabauty0` for $\mathcal{C}_7$ and $\mathcal{C}_8$, respectively.

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- Calculations in Magma show that the following points in $\mathcal{C}_6(\mathbb{Q})$ are the only ones with heights bounded by 10^8.

Lemma 8

We have
$\{(1,-4),(1,4),(-11/4,-481/64),(-11/4,481/64),\infty^+,\infty^-\}\subset \mathcal{C}_6(\mathbb{Q})$.

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- $Jac(\mathcal{C}_6)$ has $\mathbb{Q}$-rank 1 or 2 (probably 2), and a standard Chabauty's method for calculating $\mathcal{C}_6(\mathbb{Q})$ does not work. Then elliptic Chabauty method were unsuccessful.

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- By Chabauty-Coleman estimate we have $\#\mathcal{C}_6(\mathbb{Q})\leq \#\mathcal{C}_6(\mathbb{F}_5)+2=8$.

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- By Chabauty-Coleman estimate we have $\#\mathcal{C}_6(\mathbb{Q}) \leq \#\mathcal{C}_6(\mathbb{F}_5) + 2 = 8$.
- Taking into account the above estimate and calculations in Magma, we expect that in Lemma 8 we may replace $\subset$ by the equality of sets. The calculations in Magma took about 380 hours.

Remark 9

We expect (using the above Remarks) that the solutions listed in Theorem 5 are all non-trivial solutions to the Diophantine equation $x^2 + 7y^8 = 4z^3$.

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3. The sketch for the proof (Theorem 6)

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- Let us start with the conjectural description of the set of solutions of the title equation. Throughout this section we will assume that $n = p$ is a prime ≥ 7.

Conjecture 2

Let $p \geq 7$ be a prime. All non-trivial solutions to the Diophantine equation $x^2 + 7y^{2p} = 4z^3$ in coprime integers x, y, z are given by $(x, y, z) = (\pm 5, \pm 1, 2)$.

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- Conjecture 2 follows from the Conjectures 3 and 4 below. Using [Conjecture 2, Ivorra & Kraus-2006] and Proposition 13, we immediately obtain the following result.

3. The sketch for the proof (Theorem 6)

Theorem 6

Assume the *abc* conjecture. Then for a positive proportion of primes p, all non-trivial solutions to the Diophantine equation $x^2 + 7y^{2p} = 4z^3$ in coprime integers x,y,z are given by $(x, y, z) = (\pm 5, \pm 1, 2)$.

Theorem 6

Assume the *abc* conjecture. Then for a positive proportion of primes p, all non-trivial solutions to the Diophantine equation $x^2 + 7y^{2p} = 4z^3$ in coprime integers x,y,z are given by $(x, y, z) = (\pm 5, \pm 1, 2)$.

- Below we will formulate the Conjectures 3 and 4, and prove some results towards each of them.

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- Below we will formulate the Conjectures 3 and 4, and prove some results towards each of them.
- We have reduced the problem of solving the title equation to solving the equations $7\alpha^{2p} + 4\beta^p = 3u^2$ and $3^{2p-3}7\alpha^{2p} + 4\beta^p = u^2$.

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- Below we will formulate the Conjectures 3 and 4, and prove some results towards each of them.
- We have reduced the problem of solving the title equation to solving the equations $7\alpha^{2p} + 4\beta^p = 3u^2$ and $3^{2p-3}7\alpha^{2p} + 4\beta^p = u^2$.
- (i) First consider the equation

$$7\alpha^{2p} + 4\beta^p = 3u^2.$$

Conjecture 3

Let $n = p$ be a prime ≥ 7. All non-trivial solutions to the Diophantine equation $7X^{2p} + 4Y^p = 3Z^2$ in coprime odd integers X, Y, Z are given by $(X, Y, Z) = (\pm 1, -1, \pm 1)$.

- Such a conjecture, for all $p \geq p_0$, follows from a famous *abc* conjecture (see [Ivorra-Kraus, Conjecture 2-2006]).

3. The sketch for the proof (Theorem 6)

Conjecture 3

Let $n = p$ be a prime ≥ 7. All non-trivial solutions to the Diophantine equation $7X^{2p} + 4Y^p = 3Z^2$ in coprime odd integers X, Y, Z are given by $(X, Y, Z) = (\pm 1, -1, \pm 1)$.

- Such a conjecture, for all $p \geq p_0$, follows from a famous *abc* conjecture (see [Ivorra-Kraus, Conjecture 2-2006]).
- Using modular approach, we immediately obtain the following result towards Conjecture 3.

Proposition 10

Let $p \geq 7$ be a prime. The Diophantine equation $7X^{2p} + 4Y^p = 3Z^2$ has no solutions (a, b, c), where a, b, c are coprime odd integers and $b \equiv 1 \pmod 4$.

The main steps of the modular method over totally real fields can be summarized as follows.

modular approach

- Constructing a Frey curve

  Attach an elliptic curve E/K to a putative solution of a Diophantine equation, where K is some totally real field. In the case of Fermat's Last Theorem, following an idea of Frey–Hellegouarch one considers the curve

$$y^2 = x(x - a^p)(x + b^p)$$

  where $a^p + b^p = c^p$, $abc \neq 0$, $a, b, c \in \mathbb{Z}$. Studying different equations require constructing different curves; such an elliptic curve is called a Frey elliptic curve or simply Frey curve for short.

3. The sketch for the proof (Theorem 6)

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  Prove modularity of E/K.
- Irreducibility
  Prove irreducibility of $\overline{\rho}_{E,p}$, the mod p Galois representation attached to E.
- Level lowering
  Conclude that $\overline{\rho}_{E,p} \cong \overline{\rho}_{g,\mathfrak{p}}$ where g is a Hilbert newform over K of parallel weight 2, trivial character and level among finitely many explicit; here $\overline{\rho}_{g,\mathfrak{p}}$ denotes the mod $\mathfrak{p}$ representation attached to g for some $\mathfrak{p} \mid p$.

modular approach

- Contradiction
  Compute all the newforms predicted in the previous step; then, for each computed newform g and $\mathfrak{p} \mid p$ in its field of coefficients, show that $\overline{\rho}_{E,p} \not\cong \overline{\rho}_{g,\mathfrak{p}}$. This rules out the isomorphism predicted by level lowering, yielding a contradiction. This final step is also known as the elimination step.

Remark 11

Let us explain why we can't prove anything when $b \equiv 3 \pmod 4$. *One reason is that the solution* $(x, y, z) = (1, -1, 1)$ *of the equation* $7x^p + 4y^p = 3z^2$ *is going to be an obstruction, unless it corresponds to a newform with complex multiplication (as in the case of the equation* $4x^p + y^p = 3z^2$ *in section 6 of [Ivorra-Kraus-2006]). In our case we need to consider the newforms of weight* 2 *and level* 504, *and all such forms are without complex multiplication.*

- (ii) Next, we consider the equation

$$3^{2p-3}7\alpha^{2p}+4\beta^{p}=u^{2}$$

.

Conjecture 4

Let $p \geq 7$ be a prime. The Diophantine equation $3^{2p-3}7X^{2p} + 4Y^p = Z^2$ has no solutions in coprime odd integers.

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- Using modular approach, we obtain the following result towards Conjecture 4.

Proposition 12

Let $p \geq 7$ be a prime. The Diophantine equation $3^{2p-3}7X^{2p} + 4Y^p = Z^2$ has no solutions (a, b, c), where a, b, c are coprime odd integers and $b \equiv 3 \pmod 4$.

- Below we will prove some unconditional results towards Conjecture 4. As a first result we use the symplectic method to show that Conjecture 4 holds for infinitely many p's.

Proposition 13

The Diophantine equation $3^{2p-3}7X^{2p}+4Y^p=Z^2$ *has no solution in coprime odd integers for any prime* $p\equiv 5$ *or* $11 \pmod{12}$.

Symplectic method

The first symplectic criterion was established in 1992 by Kraus and Oesterlé and it is applicable when E and E' have a common prime ℓ of multiplicative reduction. The reason for the name is that the method is conceptually based on the symplectic behaviour of isomorphic Galois representations.

Symplectic/anti-symplectic isomorphism

Let $p \geq 3$ be a prime. Let E and E' be elliptic curves over $\mathbb{Q}$ and write $E[p]$ and $E'[p]$ for their p-torsion modules. Write $G_{\mathbb{Q}}$ for the absolute Galois group. Let $\varphi : E[p] \to E'[p]$ be a $G_{\mathbb{Q}}$-modules isomorphism. There is an element $d(\varphi) \in \mathbb{F}_p^{\times}$ such that, for all $P, Q \in E[p]$, the Weil pairings satisfy $e_{E',p}(\varphi(P), \varphi(Q)) = e_{E,p}(P, Q)^{d(\varphi)}$. We say that φ is a *symplectic isomorphism* if $d(\varphi)$ is a square modulo p and an *anti-symplectic* otherwise. If the Galois representation $\overline{\rho}_{E,p}$ is irreducible then all $G_{\mathbb{Q}}$-isomorphisms have the same symplectic type.

Lemma 3 (Kraus & Oesterlé, 1992)

Let $\ell \neq p$ be primes with $p \geq 3$. Let E and E' be elliptic curves over $\mathbb{Q}_\ell$ with multiplicative reduction. Suppose that $E[p]$ and $E'[p]$ are isomorphic $\mathrm{G}_{\mathbb{Q}_\ell}$-modules. Assume further that $p \nmid v_\ell(\Delta'_m)$. Furthermore

$$E[p] \text{ and } E'[p] \text{ are symplectically isomorphic } \Leftrightarrow \left(\frac{v_\ell(\Delta_m)/v_\ell(\Delta'_m)}{p}\right) = 1.$$

Moreover, $E[p]$ and $E[p]$ are not both symplectically and anti-symplectically isomorphic.

3. The sketch for the proof (Theorem 6)

- Here we will assume that $p \geq 7$ is a prime and apply variants of the method introduced by Kraus. Kraus stated a very interesting criterion [Kraus-1998] that often allows to prove that the Diophantine equation $x^3 + y^3 = z^p$ (p an odd prime) has no primitive solutions for fixed p, and verified his criterion for all primes $17 \leq p < 10^4$.

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- Such a criterion has been formulated (and refined) in other situations.

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- Such a criterion has been formulated (and refined) in other situations.

Kraus Type Criterion

Let $q \geq 11$ be a prime number, and let $k \geq 1$ be an integer factor of $q - 1$. Let $\mu_k(\mathbb{F}_q)$ denote the group of k-th roots of unity in $\mathbb{F}_q^\times$. Set

$$A_{k,q} := \{\xi \in \mu_k(\mathbb{F}_q) : \frac{7 + 2^2 3^3 \xi}{3^3} \text{ is a square in } \mathbb{F}_q\}.$$

Kraus Type Criterion

For each $\xi \in A_{k,q}$, we denote by δ_ξ the least non-negative integer such that

$$\delta_\xi^2 \mod q = \frac{7 + 2^2 3^3 \xi}{3^3}.$$

We associate with each $\xi \in A_{k,q}$ the following equation

$$Y^2 = X^3 + \delta_\xi X^2 + \xi X.$$

Its discriminant equals $2^4 3^{-3} 7 \xi^2$, so it defines an elliptic curve E_ξ over $\mathbb{F}_q$. We put $a_q(\xi) := q + 1 - \# E_\xi(\mathbb{F}_q)$.

Theorem 14 (Chałupka, Dąbrowski, Soydan-?)

Let $p \geq 7$ be a prime (resp. $p = 11$). Suppose that for each elliptic curve

$$F \in \{588C1, 1176G1\} \qquad (\textit{resp. } F \in \{168A1, 1686B1\})$$

there exists a positive integer k such that the following three conditions hold

1. $q := kp + 1$ *is a prime,*
2. $a_q(F)^2 \not\equiv 4 \pmod{p}$,
3. $a_q(F)^2 \not\equiv a_q(\xi)^2 \pmod{p}$ *for all* $\xi \in A_{k,q}$.

Then the equation $3^{2p-3}7x^{2p} + 4y^p = z^2$ has no solutions in coprime odd integers.

Corollary 15 (Chałupka, Dąbrowski, Soydan-?)

Let $7 \leq p < 10^9$ and $p \neq 13, 17$ be a prime. Then there are no triples (x, y, z) of coprime odd integers satisfying $3^{2p-3}7x^{2p} + 4y^p = z^2$.

Corollary 15 (Chałupka, Dąbrowski, Soydan-?)

Let $7 \leq p < 10^9$ and $p \neq 13, 17$ be a prime. Then there are no triples (x, y, z) of coprime odd integers satisfying $3^{2p-3}7x^{2p} + 4y^p = z^2$.

- The computations took about 270 hours (with two desktop computers).

the Selmer Chabauty method of Stoll

- The Selmer Chabauty method of Stoll [Stoll-2018] may lead to determining the set $\mathcal{D}_p(\mathbb{Q})$ for $\mathcal{D}_p : Y^2 = 12X^p + 21$, for prime values of p, $7 \leq p \leq 19$. Such a result, combined with Corollary 15 will give all solutions to the title Diophantine equations.

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- Solving the title equation can be reduced to solving the following two Diophantine equations in coprime odd integers:

$$7X^{2p} + 4Y^p = 3Z^2$$

and

$$3^{2p-3}7X^{2p} + 4Y^p = Z^2.$$

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- A Kraus type criterion shows that the second one has no solutions (see Corollary 15).

the Selmer Chabauty method of Stoll

- The first Diophantine equation leads to the curve $\mathcal{D}_p : Y^2 = 12X^p + 21$. Note that $\{\infty, (-1, 3), (-1, -3)\} \subset \mathcal{D}_p(\mathbb{Q})$, and the rational points $(-1, \pm 3)$ lead to the (obvious) solutions $(x, y, z) = (\pm 5, \pm 1, 2)$ of the Diophantine equation $x^2 + 7y^{14} = 4z^3$.

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- Let $7 \le p \le 19$ be a prime. Following the Selmer Chabauty method (here we skip some technical details) may lead to determining $\mathcal{D}_p(\mathbb{Q})$.

- We expect (compare Conjecture 2), that all non-trivial solutions to the Diophantine equation $x^2 + 7y^{10} = 4z^3$ in coprime integers x,y,z are given by $(x, y, z) = (\pm 5, \pm 1, 2)$ or $(\pm 1788379, \pm 15, 12184)$.

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- We may consider the equations

$$7\alpha^{2p} + 4\beta^p = 3u^2$$

and

$$3^{2p-3}7\alpha^{2p} + 4\beta^p = u^2$$

for $p = 5$, and in this case they lead to the genus 2 curves $\mathcal{C}_1 : Y^2 = 12X^5 + 21$ and $\mathcal{C}_2 : Y^2 = 4X^5 + 3^7 \times 7$, respectively.

- We expect (compare Conjecture 2), that all non-trivial solutions to the Diophantine equation $x^2 + 7y^{10} = 4z^3$ in coprime integers x,y,z are given by $(x, y, z) = (\pm 5, \pm 1, 2)$ or $(\pm 1788379, \pm 15, 12184)$.
- We may consider the equations

$$7\alpha^{2p} + 4\beta^p = 3u^2$$

and

$$3^{2p-3}7\alpha^{2p} + 4\beta^p = u^2$$

for $p = 5$, and in this case they lead to the genus 2 curves $\mathcal{C}_1 : Y^2 = 12X^5 + 21$ and $\mathcal{C}_2 : Y^2 = 4X^5 + 3^7 \times 7$, respectively.
- Now Magma shows that $Jac(\mathcal{C}_i)$ $(i = 1, 2)$ have $\mathbb{Q}$-ranks 2, hence a standard Chabauty's method for calculating $\mathcal{C}_i(\mathbb{Q})$ $(i = 1, 2)$ does not work.

It is easy to check the following results.

Lemma 16

We have
(i) $\{(-1,-3),(-1,3),\infty\} \subset \mathcal{C}_1(\mathbb{Q})$*;*
(ii) $\{(-5,-53),(-5,53),\infty\} \subset \mathcal{C}_2(\mathbb{Q})$*.*

- Calculations in Magma show that the above points are the only ones with heights bounded by 3×10^7 (it took about 53 hours for each curve on the desktop computer). We expect that in Lemma 16 we may replace $\subset$ by the equality of sets.

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- Calculations in Magma show that the above points are the only ones with heights bounded by 3×10^7 (it took about 53 hours for each curve on the desktop computer). We expect that in Lemma 16 we may replace $\subset$ by the equality of sets.
- As a consequence, we are lead to the above conjectural description of the set of solution to the Diophantine equation $x^2 + 7y^{10} = 4z^3$.

- Of course, we may (and will) assume that $n = p$ is a prime. Writing $7y^{2p} = (2z^6 - x)(2z^6 + x)$, we are led to consider the Diophantine equation $u^{2p} + 7v^{2p} = 4z^6$.
- Next, writing $7v^{2p} = (2z^3 - u^p)(2z^3 + u^p)$, we are led to consider the Diophantine equation

$$7V^{2p} + 2U^p = Z^{2p}. \tag{9}$$

If x, y, z in the title equation are coprime, then U, V, Z are odd and coprime.

- (i) Let $p = 2$. In this case, the title equation leads to an elliptic curve $E : Y^2 = X^3 - 28X$, with $E(\mathbb{Q}) = \{(0,0), \infty\}$.

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- (ii) Let $p = 3$. In this case, the equation $7v^6 = (2z^3 - u^3)(2z^3 + u^3)$ leads to $2z^3 = S^6 + T^3$. Now, it is well known that the only rational points on the cubic curve $X^3 + Y^3 = 2$ are $(1,1)$ and the point at infinity.

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- (ii) Let $p = 3$. In this case, the equation $7v^6 = (2z^3 - u^3)(2z^3 + u^3)$ leads to $2z^3 = S^6 + T^3$. Now, it is well known that the only rational points on the cubic curve $X^3 + Y^3 = 2$ are $(1,1)$ and the point at infinity.
- (iii) Let $p \geq 5$ be any prime. Suppose that the coprime odd integers a, b, c solve the equation (9). The associated Frey curve is

  $$E = E^p_{a,b,c} : y^2 = x(x - 7a^{2p})(x + 2b^p)$$

  (see [Kraus-1997,Section 4] for details). Here we follow the procedure of the signature (p, p, p).

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Conjecture 5

Let $p \geq 7$ be a prime. All non-trivial solutions to the Diophantine equation $x^2 + 7y^{2p} = 4z^3$ in coprime integers x,y,z are given by $(x, y, z) = (\pm 5, \pm 1, 2)$.

Conjecture 6

All non-trivial solutions to the Diophantine equation $x^2 + 7y^{10} = 4z^3$ in coprime integers x,y,z are given by $(x, y, z) = (\pm 5, \pm 1, 2)$ or $(\pm 1788379, \pm 15, 12184)$.

Question 1

Has the Diophantine equation $7x^2 + y^{2p} = 4z^3$ any solution for all primes $p \geq 10^9$?

Conjecture 7

The Diophantine equation $7x^2 + y^{14} = 4z^3$ has no solution in coprime odd integers.

- Here we discuss a few approaches to this equation and the obstacles to making them work here.

Why is hard to prove Conjecture 7 for us?

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- Here we discuss a few approaches to this equation and the obstacles to making them work here.
- (i) *The modular method*. The Diophantine equation $7x^2 + y^{14} = 4z^3$ is reduced to the equations

$$\alpha^{2p} - 4\beta^p = 21v^2$$

and

$$3^{2p-3}\alpha^{2p} - 4\beta^p = 7v^2$$

for $p = 7$: $X^{14} - 4Y^7 = 21Z^2$ and $3^{11}X^{14} - 4Y^7 = 7Z^2$, respectively. In both cases, we could not exclude the possibility that the Galois representation associated to the Frey type curve arises from newform with nonrational Fourier coefficients.

Why is hard to prove Conjecture 7 for us?

- (ii) *Chabauty type approach in genus* 3. The Diophantine equations from (i) lead to the genus 3 curves $\mathcal{D}_1 : y^2 = x^7 + 2^{12} \cdot 3^7 \cdot 7^7$ and $\mathcal{D}_2 : y^2 = x^7 + 2^{12} \cdot 3^{11} \cdot 7^7$, respectively. Magma calculations show that the only rational points on $\mathcal{D}_i(\mathbb{Q})$ (with bounds 10^9) are points at infinity, as expected. Magma also shows that ranks of $Jac(\mathcal{D}_i)(\mathbb{Q})$ $(i = 1, 2)$ are bounded by 1.

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- There are two technical problems to use Chabauty method:
  1. One needs explicit rational points of infinite order (not easy to find).
  2. There is no readily available implementation of Chabauty's method for (odd degree) hyperelliptic genus 3 curves.
- Professor Stoll suggested to try the methods of his papers [Stoll, 2018], but we were not able to follow his advise yet.

Why is hard to prove Conjecture 7 for us?

- (iii) *Combination of the modular and Chabauty methods*. One may consider a more general Diophantine equation $7x^2 + y^7 = 4z^3$, try to follow the paper [Poonen,Schaefer,Stoll-2007], and then deduce the solutions for the original Diophantine equation. It seems a very difficult task, but maybe the only available way ...

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Thank you for your attention!

Köszönöm a figyelmet!