Maximal Operators and Local Mean Value Theorems for Weyl Sums

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Set-up

Given a vector $\mathbf{u} = (u_1, \ldots, u_d) \in \mathsf{T}_d$, where

$$\mathsf{T}_d = (\mathbb{R}/\mathbb{Z})^d = d\text{-dimensional unit torus,}$$

we define our main object of study:

$$\text{Weyl Sums:} \qquad S_d(\mathbf{u}; N) = \sum_{1 \le n \le N} \mathbf{e}(u_1 n + \ldots + u_d n^d),$$

where $\mathbf{e}(x) = \exp(2\pi i x)$, named after *Hermann Weyl*, who introduced, investigated and foresaw their great value for mathematics in 1916.

As concrete examples of their capabilities, *Hermann Weyl* established:

- in 1916: the *uniformity of distribution modulo one* of the fractional parts of values of real polynomials;
- in 1921: the *subconvexity* bound for the Riemann zeta-function, the first non-trivial result towards the Lindelöf hypothesis.

Weyl sums everywhere

Since then, lots of other applications have been found, including:

- bounds on the zero-free region of $\zeta(s)$ and thus bounds for the error term in the Prime Number Theorem;
- additive problems such as the Waring problem;
- bounds on very short character sums and thus on the L-functions with highly composite moduli;
- low-lying zeros of families of L-functions of elliptic curves;
- various problems from the uniformity of distribution theory and Diophantine approximations;
- Large sieve inequalities for polynomial moduli;

$\vdots$

Later we will also mention some surprising applications to PDE's:
M. B. Erdogan & G. Shakan (**2019**).

What do we know about Weyl sums?

Average values: Trivially, by the *Parseval identity*,

$$\int_{\mathsf{T}_d} |S_d(\mathbf{u}; N)|^2 \mathrm{d}\mathbf{u} = N.$$

Bounds on higher moments

$$J_{d,s}(N) = \int_{\mathsf{T}_d} |S_d(\mathbf{u}; N)|^{2s} \mathrm{d}\mathbf{u}, \qquad s = 2, 3, \ldots,$$

are highly nontrivial if $s > d$, for $s \le d$ the reasonably elementary method of *Mordell* (**1932**) works. They are known under the collective name:

Vinogradov's Mean Value Theorem{s} (**VMVT**)

I. M. Vinogradov (**1935**):

(i) obtained the first nontrivial bounds on $J_{d,s}(N)$ with a *"right saving"* (but for larger than really necessary values of s);

(ii) linked such average bounds to **pointwise** bounds on $|S_d(\mathbf{u}; N)|$.

After works of *I. M. Vinogradov; Yu. V. Linnik; N. M. Korobov; A. A. Karatsuba; K. Ford; R. C. Vaughan; T. Wooley; . . .* , 85 years and several dozens of papers later, we have the following:

Optimal VMVT — Bourgain, Demeter & Guth; Wooley (2016–2019)

For $s = 2, 3, \ldots$, we have

$$N^s + N^{2s-d(d+1)/2} \ll J_{d,s}(N) \ll N^{s+o(1)} + N^{2s-d(d+1)/2}.$$

This is due to

- *T. Wooley* (**2016**) for $d = 3$;
- *J. Bourgain, C. Demeter & L. Guth* (**2016**) for $d \geq 4$;
- *T. Wooley* (**2019**) for more general exponential sums with $\mathbf{e}\left(u_1\varphi_1(n) + \ldots + u_d\varphi_d(n)\right)$, $\varphi_j \in \mathbb{Z}[T]$, $j = 1, \ldots, d$.

Remark:

The upper bound is equivalent to the estimate

$$J_{d,d(d+1)/2}(N) \leq N^{d(d+1)/2+o(1)}$$

for the critical value $s = d(d+1)/2$.

Pointwise bounds: Here our knowledge is quite scarce.

Vinogradov's Method + Optimal **VMVT**

Let $\mathbf{u} = (u_1, \ldots, u_d) \in \mathsf{T}_d$ *be such that for some* ν *with* $2 \le \nu \le d$ *and some integers* a *and* q *with* $\gcd(a, q) = 1$ *we have*

$$|u_\nu - a/q| \le 1/q^2.$$

Then

$$|S_d(\mathbf{u}; N)| \le N^{1+o(1)} \left(q^{-1} + N^{-1} + qN^{-\nu}\right)^{1/d(d-1)}.$$

Nowadays we do not have any plausible approach to do better, eg, replace $1/d(d-1) \to 1/d$ and drop N^{-1}, as expected.

Remark:

It is obvious that any bound of this kind must depend on Diophantine properties of the non-linear coefficients $u_2, \ldots, u_d$.

What is next?

Recall that for the **Weyl sums**:

- We have a complete knowledge of their **average** values.
- We *know something but overall very little* about their **pointwise** behaviour.

Question:

Can we "interpolate" between these two types of results?

This question leads us to studying some very well-known notions of Functional Analysis:

Maximal Operators and Restriction Bounds

for the **Weyl sums** $S_d(\mathbf{u}; N)$ (as functions of $\mathbf{u}$).

Recall:

Let $\mathbf{u} = (u_1, \ldots, u_d) \in \mathsf{T}_d$ be such that for some ν with $2 \le \nu \le d$ and some positive integers a and q with $\gcd(a, q) = 1$ we have

$$|u_\nu - a/q| \le 1/q^2.$$

Then

$$|S_d(\mathbf{u}; N)| \le N^{1+o(1)} \left(q^{-1} + N^{-1} + qN^{-\nu}\right)^{1/d(d-1)}.$$

Observation: The bound depends on approximations to only **one** of the non-linear coefficients $u_2, \ldots, u_d$, say, u_d and for **a.a.** $u_d \in [0, 1]$ we can choose $q = N^{1+o(1)}$ in the above.

Hence, we immediate derive

For **a.a.** $u_d \in [0, 1]$ and **all** $(u_1, \ldots, u_{d-1}) \in \mathsf{T}_{d-1}$, we have

$$|S_d(\mathbf{u}; N)| \le N^{1-1/d(d-1)+o(1)}, \qquad \text{as } N \to \infty.$$

For **a.a.** $u_d \in [0,1]$ and **all** $(u_1, \ldots, u_{d-1}) \in \mathsf{T}_{d-1}$, we have

$$|S_d(\mathbf{u}; N)| \leq N^{1-1/d(d-1)+o(1)}, \qquad \text{as } N \to \infty.$$

Question:

Can we have stronger and/or more general statements?

Prototype Theorem

*For **a.a.** components of $\mathbf{u} \in \mathsf{T}_d$ on prescribed k positions the following holds: For **all** components on the remaining $d-k$ positions, for **all** $N \in \mathbb{N}$, we have $|S_d(\mathbf{u}; N)| \leq XXX$ (whatever we can prove for XXX).*

This has been studied, with a chain of consecutive improvements, by:

L. Flaminio & G. Forni (**2014**);
T. Wooley (**2016**);
C. Chen & I.S. (**2019**).

It is convenient to reformulate and generalise this question.

Given

- a vector $\boldsymbol{\varphi} = (\varphi_1, \ldots, \varphi_d) \in \mathbb{Z}[T]^d$ of d linearly independent with constants polynomials,
- a vector $\mathbf{u} = (u_1, \ldots, u_d) \in \mathsf{T}_d$,

we define

$$T_{\boldsymbol{\varphi}}(\mathbf{u}; N) = \sum_{n=1}^{N} \mathbf{e}\left(u_1\varphi_1(n) + \ldots + u_d\varphi_d(n)\right).$$

For $\varphi_i(T) = T^i$ these are **Weyl sums** $S_d(\mathbf{u}; N)$. Decompose $\mathbf{u} \in \mathsf{T}_d = \mathsf{T}_k \times \mathsf{T}_{d-k}$ as

$$\mathbf{u} = (\mathbf{x}|\mathbf{y}) \in \mathsf{T}_k \times \mathsf{T}_{d-k}$$

and write

$$T_{\boldsymbol{\varphi}}(\mathbf{x}, \mathbf{y}; N) = T_{\boldsymbol{\varphi}}(\mathbf{u}; N).$$

We emphasise that $\boldsymbol{\varphi}$ is a **vector** rather than a set — the order matters!

Maximal Operators on Weyl Sums

Following *L. Flaminio & G. Forni* (**2014**), *T. Wooley* (**2016**), we are interested in bounds on

$$T_{\boldsymbol{\varphi}}(\mathbf{x}, \mathbf{y}; N)$$

which hold for

a.a. $\mathbf{x} \in \mathsf{T}_k$ and **all** $\mathbf{y} \in \mathsf{T}_{d-k}$.

Equivalently, we are interested in bounds on

Maximal Operators: $$\sup_{\mathbf{y} \in \mathsf{T}_{d-k}} |T_{\boldsymbol{\varphi}}(\mathbf{x}, \mathbf{y}; N)|$$

which hold for **a.a.** $\mathbf{x} \in \mathsf{T}_k$.

Why do we expect $\sup_{\mathbf{y}\in\mathsf{T}_{d-k}} |T_\varphi(\mathbf{x},\mathbf{y};N)|$ to be small?

The set of large **Weyl sums** is very *sparse*.

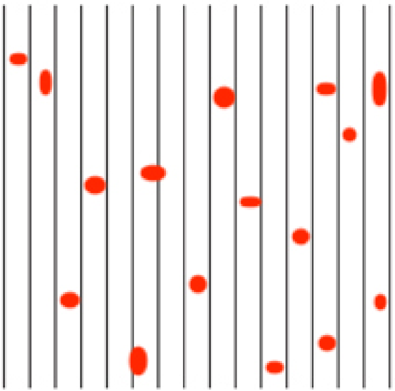

Figure: Almost all vertical lines miss red areas ● of large **Weyl sums** in T_2

Some concrete results

L. Flaminio & G. Forni (**2014**);
T. Wooley (**2016**);
C. Chen & I.S. (**2019**):

For φ with a nontrivial Wronskian, for **a.a.** $\mathbf{x} \in \mathsf{T}_k$,

$$\sup_{\mathbf{y} \in \mathsf{T}_{d-k}} |T_\varphi(\mathbf{x}, \mathbf{y}; N)| \leqslant N^{1/2+\gamma+o(1)}, \qquad N \to \infty,$$

with some $\gamma < 1/2$.

To formulate concrete results we need the following **important parameter:**

$$\sigma_k(\boldsymbol{\varphi}) = \sum_{j=k+1}^{d} \deg \varphi_j$$

$=$ sum of degrees in the $\mathbf{y}$-part over which we maximise.

Wooley (2016)

For $1 \le k \le d-1$ we can take

$$\gamma_W = \frac{2\sigma_k(\boldsymbol{\varphi}) + d - k + 1}{2d^2 + 4d - 2k + 2}.$$

Using completing technique and a new self-improving argument:

Chen & Shparlinski (2019)

For $1 \le k \le d-1$ we can take

$$\gamma_{CS} = \frac{2\sigma_k(\boldsymbol{\varphi}) + d - k}{2d^2 + 4d - 2k} < \gamma_W.$$

Remark

This is nontrivial, ie, $\gamma_{CS} < 1/2$ **iff**

$$\sigma_k(\boldsymbol{\varphi}) < d(d+1)/2,$$

which always holds in the classical case

$$\{\varphi_1(T), \ldots, \varphi_d(T)\} = \{T, \ldots, T^d\}$$

but may fail otherwise, eg, take $d = 2$ *and* $\boldsymbol{\varphi} = (T, T^m)$, $m \ge 3$.

Remark

*C. Chen & I.S. (**2019**): For*

$$\{\varphi_1(T), \ldots, \varphi_d(T)\} = \{T, \ldots, T^d\}$$

and $k = d$ (ie, without $\sup$) we can take $\gamma = 0$. This recovers the well-known statement that for **a.a.** $\mathbf{u} \in \mathsf{T}_d$,

$$|S_d(\mathbf{u}; N)| \le N^{1/2+o(1)}, \qquad N \to \infty.$$

Question: (Should we always expect square-root cancellation?)

Can we can take $\gamma = 0$ for "generic enough" $\boldsymbol{\varphi}$, eg, $\boldsymbol{\varphi} = \left(T, \ldots, T^d\right)$?

We believe this is false and in some cases we can prove that $\gamma \ge 1/4$.

Norms of Maximal Operators

Maximal Operators are well-known in Functional Analysis:

$$\mathsf{M}_k : \quad F(\mathbf{x},\mathbf{y}) \mapsto G(\mathbf{x}) = \sup_{\mathbf{y}\in\mathsf{T}_{d-k}} |F(\mathbf{x},\mathbf{y})|$$

We have discussed bounds on $\mathsf{M}_k\left(T_{\boldsymbol{\varphi}}(\mathbf{x},\mathbf{y};N)\right)$ for **a.a.** $\mathbf{x}\in\mathsf{T}_k$.

A variation of this is a question about bounds on the L^ρ-norm:

$$\left\|\mathsf{M}_k\left(T_{\boldsymbol{\varphi}}(\mathbf{x},\mathbf{y};N)\right)\right\|_\rho = \left(\int_{\mathsf{T}_k} \mathsf{M}_k\left(T_{\boldsymbol{\varphi}}(\mathbf{x},\mathbf{y};N)\right)^\rho \, d\mathbf{x}\right)^{1/\rho}.$$

To simplify the discussion from now on we always assume that $\varphi_i(T) = T^i$, $i = 1,\ldots,d$, and thus we look at

$$\mathsf{M}_k\left(S_d(\mathbf{x},\mathbf{y};N)\right) = \sup_{\mathbf{y}\in\mathsf{T}_{d-k}} \left|S_d(\mathbf{x},\mathbf{y};N)\right|.$$

Baker, Chen & Shparlinski (2021)

For any positive $\rho \geq d^2 + 2d - k$, for $N \to \infty$, we have

$$N^{1-k(k+1)/2\rho} \ll \left\| \mathsf{M}_k \left(S_d(\mathbf{x}, \mathbf{y}; N) \right) \right\|_\rho \leq N^{1-k(k+1)/2\rho + o(1)}.$$

Remark

The significance of the cut-off $d^2 + 2d - k$ is in this interpretation:

$$\begin{aligned} d^2 + 2d - k &= d(d+1) + d - k \\ &= 2 \times \textit{critical exponent in } \textbf{VMVT} + \textit{dimension of } \mathbf{y} \textit{ in } \sup. \end{aligned}$$

Remark

*By convexity, we can also have an upper bound for $\rho < d^2 + 2d - k$ and recover the previous result of C. Chen & I.S. (**2019**).*

For $d = 2$, ie, for the maximal operator on Gauss sums

$$M_1(G(x,y)) = \sup_{y\in[0,1]} |G(x,y)|, \quad \text{where} \quad G(x,y) = \sum_{n=1}^{N} \mathbf{e}(xn + yn^2),$$

R. Baker (**2021**), refining a result of *A. Barron* (**2020**), has given matching upper and lower bounds:

Baker (2021)

We have

$$N^{a(\rho)}(\log N)^{b(\rho)} \ll \|M_1(G(x,y))\|_\rho \ll N^{a(\rho)}(\log N)^{b(\rho)},$$

where

$$a(\rho) = \begin{cases} 3/4 & \text{for } 1 \le \rho \le 4, \\ 1 - 1/\rho & \text{for } \rho > 4, \end{cases} \qquad b(\rho) = \begin{cases} 1/\rho & \text{for } \rho = 4, \\ 0 & \text{for } \rho \ge 1,\ \rho \ne 4. \end{cases}$$

Question:

Can we extend this to any $d \ge 2$ and control $\|\mathsf{M}_k(S_d(\mathbf{x},\mathbf{y};N))\|_\rho$ for any $\rho \ge 1$ rather than only for $\rho \ge d^2 + 2d - k$?

Binomial Weyl Sums and PDE's

Let Ω_m be the smallest possible value of ϑ such that for any $\varphi(T) \in \mathbb{Z}[T]$ of degree m and any $\tau \in \mathbb{R}$ for **a.a.** $x \in [0,1]$ we have

$$\sup_{y\in[0,1]} \left| \sum_{n=1}^{N} \mathbf{e}\left(x\varphi(n) + y(\tau\varphi(n) + n)\right) \right| \le N^{\vartheta+o(1)}.$$

This is exactly the previous scenario of maximal operators of *L. Flaminio & G. Forni* (**2014**), *T. Wooley* (**2016**) and *C. Chen & I.S.* (**2019**).

- *Bad news*: none of the previous bounds works;
- Good news: but the methods do!

Remark

*These sums look weird but their existence is justified by applications to Schrödinger, Korteweg-de Vries, Airy and other classical PDE's, see M. B. Erdogan & G. Shakan (**2019**) — fractal dimension of solutions.*

M. B. Erdogan & G. Shakan (**2019**):

$$\Omega_m \le \min\left\{1 - \frac{1}{2^m + 1}, 1 - \frac{1}{2m(m-1)+1}\right\}.$$

The results of *C. Chen & I.S.* (**2019**) on "dense" **Weyl sums** do not work but the method does:

Chen & Shparlinski (2019)

We have

$$\Omega_m \le 1 - \frac{1}{2s(m)+1},$$

where

$$s(2) = 3, \quad s(3) = 5, \quad s(4) = 8, \quad s(5) = 12, \quad s(6) = 18,$$
$$s(7) = 24, \quad s(8) = 31, \quad s(9) = 40, \quad s(10) = 49,$$

while for $m \ge 11$ *we define* $r(m) = \left\lfloor \sqrt{2m+2} \right\rfloor$

$$s(m) = m(m-1)/2 + r(m) - \begin{cases} 0, & 2m+2 \ge r(m)^2 + r(m), \\ 1, & \text{otherwise.} \end{cases}$$

What is truth about Ω_m?

J. Brandes, S. T. Parsell, C. Poulias, G. Shakan & R. C. Vaughan (**2020**):

$$\Omega_2 = \Omega_3 = 3/4.$$

J. Brandes & I.S. (**2020**): For any $m \geq 2$, we have $\Omega_m \geq 3/4$.

Question:

Is it true that for any $m \geq 2$ we have $\Omega_m = 3/4$?

Remark

*For $m = 4$ the upper bound $\Omega_4 \leq 16/17$, due to C. Chen & I.S. (**2019**), is already very far.*

Question:

What about sums with two nonlinear polynomials, eg,

$$\sup_{y \in [0,1]} \left| \sum_{n=1}^{N} \mathbf{e}\left(x\varphi(n) + y(\tau\varphi(n) + n^2)\right) \right| ?$$

Local Mean Value Theorems for Weyl Sums

Let μ be a measure supported on some set $\mathcal{V} \subseteq \mathsf{T}_d$, thus $\mu(\mathcal{V}) = 1$. Our goal here is to estimate the following mean values on $\mathcal{V}$:

VMVT Restricted to $\mathcal{V}$: $$\int_{\mathcal{V}} |S_d(\mathbf{u}; N)|^\rho \mathrm{d}\mu(\mathbf{u}).$$

The set $\mathcal{V}$ can be some

- algebraic structure, eg, an algebraic variety;
- analytic structure, eg, a smooth curve or a surface defined by analytic functions;
- geometric structure, eg, a linear space or an intersection of spheres, balls and convex bodies;
- combinatorial structure, eg, a sets with a small sumset and a generalised arithmetic progression.

Warning: When $\mathcal{V}$ shrinks, eg, becomes a small box, we are approaching the scenario of pointwise bounds.

Small boxes

$$\mathcal{C}_{\boldsymbol{\xi},\delta} = [\xi_1, \xi_1 + \delta] \times \cdots \times [\xi_d, \xi_d + \delta].$$

In fact, this case has applications to most of the other cases and to several other problems. For example, *C. Chen, B. Kerr, J. Maynard & I.S.* (**2020**), established the optimal for $\delta \gg N^{-1/2}$ bound

$$\int_{\mathcal{C}_{\boldsymbol{\xi},\delta}} |S_d(\mathbf{u}; N)|^4 \mathrm{d}\mathbf{u} \ll \delta^d N^2 + \delta^{d-2} N^{1+o(1)}$$

and used it in studying the Lebesque measure of the set of **Weyl sums** with exactly square root cancellation, ie, with

$$cN^{1/2} \le |S_d(\mathbf{u}; N)| \le CN^{1/2}.$$

Also, various bounds can be found in
C. Chen & I.S. (**2019**),
C. Demeter & B. Langowski (**2021**),
C. Chen, J. Brandes & I.S. (**2023**).

The worst local MVT

For $s > 0$, we define

$$I_{s,d}(\delta; N) = \sup_{\boldsymbol{\xi} \in \mathsf{T}_d} \int_{\mathcal{C}_{\boldsymbol{\xi},\delta}} |S_d(\mathbf{u}; N)|^{2s} \mathrm{d}\mathbf{u}.$$

There is a huge zoo of bounds and conjectures. Here are some plots where we set

$$\delta = N^{-\tau} \quad \text{and} \quad \kappa_{s,d}(\tau) = \limsup_{N \to \infty} \frac{\log I_{s,d}(N^{-\tau}; N)}{\log N},$$

The lowest plot wins $\kappa_{2,2}(\tau)$

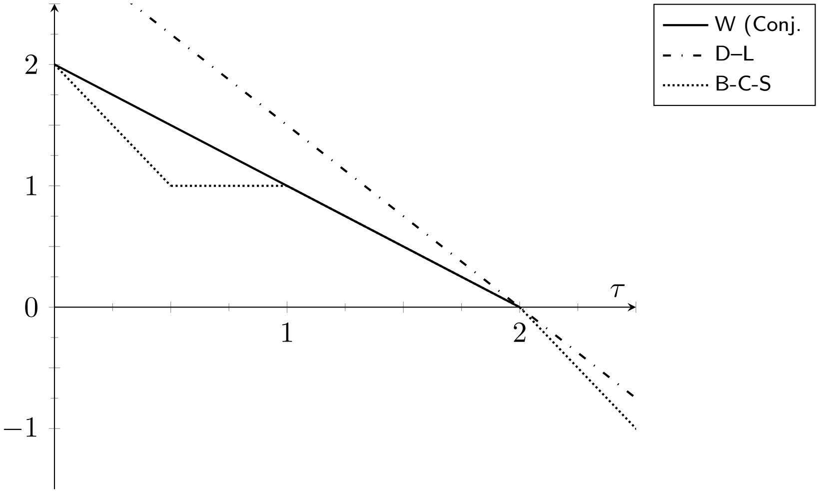

Figure: $d = s = 2$, D-L = *C. Demeter & B. Langowski* (**2021**), W = *Wooley* (**2023**), B-C-S = *C. Chen, J. Brandes & I.S.* (**2023**)

The lowest plot wins: $\kappa_{3,3}(\tau)$

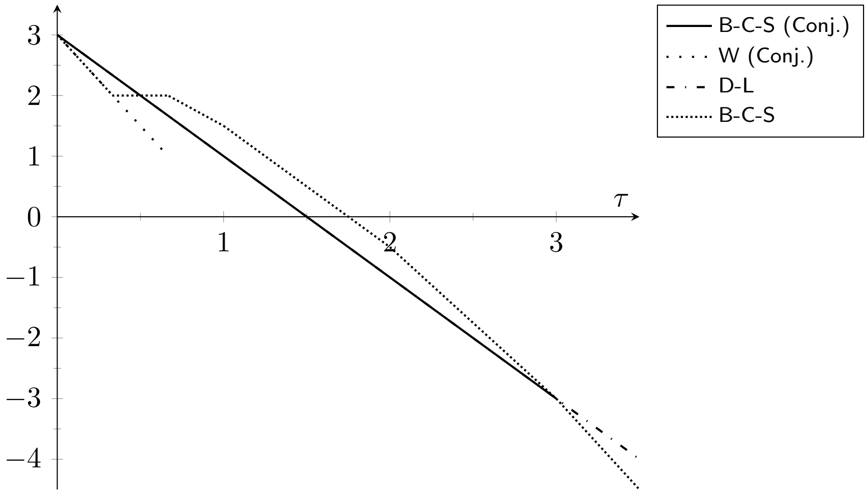

Figure: $d = s = 3$, D-L = *C. Demeter & B. Langowski* (**2021**), W = *Wooley* (**2023**), B-C-S = *C. Chen, J. Brandes & I.S.* (**2023**)

The lowest plot wins: $\kappa_{2,3}(\tau)$

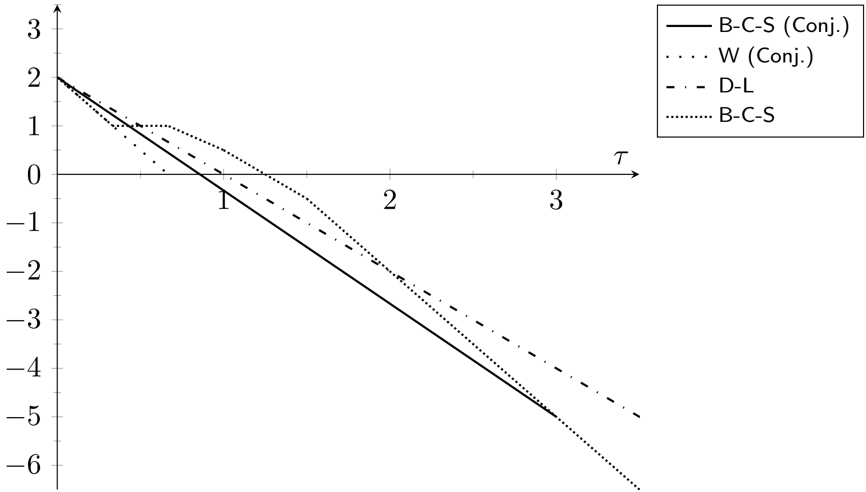

Figure: $d = 3$, $s = 2$, D-L = *C. Demeter & B. Langowski* (**2021**), W = *Wooley* (**2023**), B-C-S = *C. Chen, J. Brandes & I.S.* (**2023**)

Ideas behind the proofs

In *C. Chen, J. Brandes & I.S.* (**2023**) we improve some results of *C. Demeter & B. Langowski* (**2021**) on $I_{s,d}(\delta; N)$ using a combination of two different approaches:

- Results of *R. Baker* (**1981**) on the structure of large **Weyl sums**.
- Bounds on complete rational exponential sums

$$\left|\sum_{x=1}^{q} \mathbf{e}(F(x)/q)\right| \le q^{o(1)} \prod_{i=2}^{d} q_i^{1-1/i}, \qquad F \in \mathbb{Z}[X],\ \deg F = d$$

  which depend on the arithmetic structure of $q = q_2 \ldots q_d$ with $\gcd(q_i, q_j) = 1$ for $2 \le i < j \le d$, such that
  (i) q_2 is cube-free,
  (ii) q_i is i-th power-full but $(i+1)$-th power-free when $3 \le i \le d-1$,
  (iii) q_d is d-th power-full.
- Results on the "inhomogeneous" **VMVT**: due to *J. Brandes & K. Hughes* (**2021**) and *T. Wooley* (**2022**)

Structure of large Weyl sums

We make use of the following result (refined major arcs):

Baker (1981)

We fix some $\varepsilon > 0$ *and suppose that for a real*

$$A > N^{1-1/2d(d-1)+\varepsilon}$$

we have

$$|S_d(\mathbf{u}; N)| \ge A.$$

Then there exist integers $q, r_1, \ldots, r_d$ *such that*

$$1 \le q \le \left(NA^{-1}\right)^d N^{\varepsilon}, \qquad \gcd(q, r_1, \ldots, r_d) = 1$$

and

$$\left| u_j - \frac{r_j}{q} \right| \le q^{-1} \left(NA^{-1}\right)^d N^{-j+\varepsilon}, \qquad j = 1, \ldots, d.$$

Inhomogeneous VMVT

The classical form of the **VMVT** gives a precise bound on the number of solutions to the system of equations

$$x_1^i + \ldots + x_s^i = x_{s+1}^i + \ldots + x_{2s}^i, \quad i = 1, \ldots, d,$$
$$1 \le x_1, \ldots, x_{2s} \le N.$$

Any such bound implies the same bound for the inhomogeneous system

$$x_1^i + \ldots + x_s^i = x_{s+1}^i + \ldots + x_{2s}^i + h_i, \quad i = 1, \ldots, d,$$
$$1 \le x_1, \ldots, x_{2s} \le N.$$

However, if $(h_1, \ldots, h_d) \neq \mathbf{0}$ then we can hope for a better bound because the case $\{x_1, \ldots, x_s\} = \{x_{s+1}, \ldots, x_{2s}\}$ does not contribute anymore.

R. Baker, M. Munsch & IS (**2021**):
Besides the above application to the **VMVT** over a small cube such bounds also relevant to large sieve estimates over polynomial moduli.

J. Brandes & K. Hughes (**2021**) and *T. Wooley* (**2022**) give such better bounds, but the **truth** is not clear yet.

The existence of large complete rational sums

Our lower bounds rest on the following estimate for the complete sums

$$S_\varphi(q; a, c) = \sum_{x=1}^{q} \mathbf{e}\left((ax + c\varphi(x))/q\right)$$

with $\varphi(X) = a_d X^d + \ldots + a_1 X$.

Brandes & Shparlinski (2020)

Let p be a prime satisfying $p > (2k)^4$ with $p \nmid a_d$, and let $c \in \mathbb{Z}$ with $p \nmid c$. Then there exists $a \in \mathbb{Z}$ with $p \nmid (a + c)$ such that

$$|S_\varphi(p; a, a + c)| \geq 0.3p^{1/2}.$$

The proof is based on the bound of *E. Bombieri* (**1966**) for exponential sums along an algebraic curve over $\mathbb{F}_p$.

We combine it with an approximation formula of *R. C. Vaughan* (**1997**) and a result of *R. J. Duffin & A. C. Schaeffer* (**1941**) on approximation of almost all real numbers by fractions with prime denominators.

Further Extensions and Generalisation

Question:

Extend the range of sets of polynomials $\boldsymbol{\varphi} = (\varphi_1, \ldots, \varphi_d) \in \mathbb{Z}[T]^d$ which admit non-trivial bounds on **maximal operators***.*

Remark:

We need good versions of **VMVT** *with $\boldsymbol{\varphi}$. One interesting example is provided by Bourgain (***2017***):*

$$\int\int_{(x,y)\in \mathsf{T}_2} \left| \sum_{n=1}^{N} \mathbf{e}(xn^2 + yn^4) \right|^{10} dxdy \le N^{17/3+o(1)}.$$

Question:

What are "correct" multidimensional analogues for the sums

$$\sum_{n_1=1}^{N_1} \ldots \sum_{n_s=1}^{N_m} \mathbf{e}\left(u_1\varphi_1(n_1,\ldots,n_s) + \ldots + u_d\varphi_d(n_1,\ldots,n_s)\right)$$

with $(\varphi_1,\ldots,\varphi_d) \in \mathbb{Z}[T_1,\ldots,T_s]^d$*?*

Remark:

Some versions of the **VMVT** *are known, S. Parsell, S. Prendiville & T. Wooley (**2013**), S. Guo & R. Zhang (**2019**), S Guo (**2020**), but not in the same generality as in the one-dimensional case; many other tools are also missing.*

Definition

The discrepancy of $\boldsymbol{\gamma}_n = (\gamma_{1,n}, \ldots, \gamma_{d,n}) \in \mathsf{T}_d$, $n = 1, \ldots, N$ *is defined as*

$$D_N = \sup_{\mathfrak{B} \subseteq \mathsf{T}_d} \left| \#\{1 \le n \le N : \ \boldsymbol{\gamma}_n \in \mathfrak{B}\} - \mathrm{vol}(\mathfrak{B}) N \right|$$

where $\mathfrak{B} = [\alpha_1, \beta_1] \times \ldots \times [\alpha_d, \beta_d] \subseteq \mathsf{T}_d$ *is a box of volume* $\mathrm{vol}(\mathfrak{B}) = (\beta_1 - \alpha_1) \ldots (\beta_d - \alpha_d)$.

For $\boldsymbol{\varphi} = (\varphi_1, \ldots, \varphi_d) \in \mathbb{Z}[T]^d$ and $\mathbf{u} = (u_1, \ldots, u_d) \in \mathsf{T}_d$, let $D_{\boldsymbol{\varphi}}(\mathbf{u}; N)$ be the discrepancy of

$$(\{u_1\varphi_1(n)\}, \ldots \{u_d\varphi_d(n)\}), \qquad n = 1, \ldots, N.$$

As before, we decompose $\mathbf{u} = (\mathbf{x}|\mathbf{y}) \in \mathsf{T}_k \times \mathsf{T}_{d-k}$ and write

$$D_{\boldsymbol{\varphi}}(\mathbf{x}, \mathbf{y}; N) = D_{\boldsymbol{\varphi}}(\mathbf{u}; N).$$

Question:

Estimate

$$\mathsf{M}_k\left(D_{\boldsymbol{\varphi}}(\mathbf{x},\mathbf{y};N)\right) = \sup_{\mathbf{y}\in\mathsf{T}_{d-k}} \left|D_{\boldsymbol{\varphi}}(\mathbf{x},\mathbf{y};N)\right|.$$

for **a.a.** $\mathbf{x}\in\mathsf{T}_k$ *and on average with respect to the* L^ρ*-norm:*

$$\left\|\mathsf{M}_k\left(D_{\boldsymbol{\varphi}}(\mathbf{x},\mathbf{y};N)\right)\right\|_\rho = \left(\int_{\mathsf{T}_k} \mathsf{M}_k\left(D_{\boldsymbol{\varphi}}(\mathbf{x},\mathbf{y};N)\right)^\rho d\mathbf{x}\right)^{1/\rho}.$$

Remark:

By the **Koksma–Szüsz inequality**, *we can express* $D_{\boldsymbol{\varphi}}(\mathbf{u};N)$ *via certain linear combinations of* **Weyl sums**. *The previous methods should work with some modifications and adjustments.*

Thank you!!

Questions *and especially* **Answers** *are very We***y***lcome*