

# Recent Advances on the number of solutions to the ternary purely exponential Diophantine equation

$$a^x + b^y = c^z$$

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Number of solutions in positive integers  $x, y, z$  to

$$a^x + b^y = c^z$$

for  $\min(a, b, c) > 1$

Pillai case ( $y_1 = y_2$ )

$$a^x + b = c^z$$

in positive integers  $x, z$ .

# Cases to be Solved

In counting the number of solutions  $N(a, b, c)$  we consider two solutions  $(x_1, y_1, z_1), (x_2, y_2, z_2)$  the same solution if  $\{a^{x_1}, b^{y_1}\} = \{a^{x_2}, b^{y_2}\}$ .

*atmost2Pillai*,  $\gcd=1$

*atmost2Pillai*,  $\gcd>1$

*atmost2*,  $\gcd=1, 2 \nmid c$

*atmost2*,  $\gcd=1, 2 \mid c, \max(a, b, c) > 10^{62}$

*atmost2*,  $\gcd=1, 2 \mid c, \max(a, b, c) \leq 10^{62}$

*atmost2*,  $\gcd>1$

*atmost1Pillai*,  $\gcd=1$

*atmost1Pillai*,  $\gcd>1$

*atmost1*,  $\gcd=1$

*atmost1*,  $\gcd>1$

*atmost2Pillai*,  $\gcd=1$

Bennett (2001)

using lower bounds in linear forms in logarithms

*atmost2Pillai*,  $\gcd>1$

Bennett (2001)

elementary

at most 2 for  $a^x + b^y = c^z$ ,  $\gcd(a, b) = 1$

Best available published bound before recent work:  $N(a, b, c) < 2^{36}$   
(derived from Beukers and Schlickewei, 1996).

Unpublished bound:  $N(a, b, c) < 200$  (Hirata-Kohno)

Recent results showing  $N(a, b, c) \leq 2$ :

- $c$  odd (Scott and Styer. Number of Solutions to  $a^x + b^y = c^z$ , Debrecen, 2016.)
- $c$  even and  $\max(a, b, c) > 10^{62}$  (Hu and Le. An upper bound for the number of solutions of ternary purely exponential Diophantine equations, II, Debrecen 2019.)
- $c$  even and  $\max(a, b, c) \leq 10^{62}$  (Miyazaki and Pink. Number of solutions to a special type of unit equation in two variables, Amer. J. Math. (2024))

atmost2 for  $a^x + b^y = c^z$ ,  $\gcd(a, b) = 1$ , for  $2 \nmid c$

$$a^x + b^y = c^z, 2 \nmid c. \quad (1)$$

Letting  $D$  be the squarefree number such that  $(\frac{a^x b^y}{D})^{1/2} \in \mathbb{Z}$ , we derive the following equations in ideals in  $\mathbb{Q}(\sqrt{-D})$ :

$$[a^x - b^y + 2\sqrt{-a^x b^y}] = \mathfrak{c}^{2z}, [a^x - b^y - 2\sqrt{-a^x b^y}] = \overline{\mathfrak{c}}^{2z}, \quad (2)$$

where  $\mathfrak{c}\overline{\mathfrak{c}} = [c]$  is a factorization of  $[c]$  into conjugate ideals in  $\mathbb{Q}(\sqrt{-D})$ .

Basic Theorem: There is at most one solution  $(x, y, z)$  to (2) for a given ideal factorization  $\mathfrak{c}\overline{\mathfrak{c}}$  in  $\mathbb{Q}(\sqrt{-D})$ .

$D$  is determined by the number of parity classes for  $(x, y)$ . The number of ideal factorizations  $\mathfrak{c}\overline{\mathfrak{c}}$  for a given  $D$  is determined by  $\omega$ , the number of primes dividing  $c$ . For (1), the number of parity classes for  $(x, y)$  is four, and the number of ideal factorizations of  $[c]$  for a given  $D$  is  $2^{\omega-1}$ . This gives (Le, 1999)

$$N(a, b, c) \leq 2^{\omega+1}.$$

The key step used to improve the bound  $N(a, b, c)$  (and also improve the absolute bound  $N(a, b, c) \leq 2^{36}$  found by Hirata-Kohno, who later announced  $N(a, b, c) \leq 200$ ) is finding that there is an inverse proportionality relationship between  $p$  and  $q$ , where  $p$  is the number of parity classes for  $(x, y)$  which are possible in the congruence  $a^x + b^y \equiv 0 \pmod{c}$ , and  $q$  is the number of ideal factorizations  $\mathfrak{c}\bar{\mathfrak{c}}$  for a given  $D$  which can occur in (2).

For the simple case  $a^x + b^y = c^z$ ,  $2 \nmid c$ , the use of this idea gives just two cases:

|         | $p$ | $q$ |
|---------|-----|-----|
| Case 1: | 2   | 1   |
| Case 2: | 1   | 2   |

However, the use of the inverse proportionality is useful in handling more general equations, leading to results such as the following (Scott-Styer, Debrecen 2018):

## Theorem

Let  $n \geq 1$  and let  $c, d_1, \dots, d_n$  be integers greater than 1 such that  $c$  is odd and  $d_1, \dots, d_n$  are coprime with  $c$ . Let  $N$  be the number of solutions  $(X, Y, z)$  to the equation

$$X + Y = c^z,$$

where  $z \in \mathbb{Z}$ ,  $X = \prod_{i=1}^n d_i^{x_i}$ ,  $Y = \prod_{i=1}^n d_i^{y_i}$ ,  $x_i, y_i \in \mathbb{Z}$ ,  $\max(x_i, y_i) > 0$ ,  $\min(x_i, y_i) = 0$ , and  $X < Y$ .

Then  $N \leq 2^{n-1} + 1$ .

This result follows from proving  $pq = 2^{n-1}$ ; the '+1' handles the possibility of a single exceptional case.

$c$  even and  $\max\{a, b, c\} > 10^{62}$  (Hu and Le, 2019)

Earlier result of Hu and Le:

$$\max\{x, y, z\} < 6500(\log(\max\{a, b, c\}))^3$$

linear forms in logs (Laurent, Mignotte, Nesterenko, 1995)

linear forms in  $p$ -adic logs (Bugeaud, 1999)

This bound later used by Miyazaki and Pink.

Later improved (exponent 3 replaced by exponent  $2 + \epsilon$ ), Subburam (2018).

Now elementary approaches suffice, using  $c$  even:

Their Lemma 4.6 bounds  $\max\{a, b, c\}$  when 3 solutions satisfying certain conditions exist.

When Lemma 4.6 does not apply, continued fractions are derived from the exponents  $x_i, y_i, z_i$  which lead to an upper bound on  $\max\{a, b, c\}$ .

Two main steps:

- Results of Hu and Le (2009) used and improved to reduce bound on  $\max(a, b, c)$  when 3 solutions exist.
- Assuming 3 solutions,  $(x_i, y_i, z_i)$ ,  $i = 1, 2, 3$ , with  $z_1 \leq z_2 \leq z_3$ , sharp bounds on  $x_1, y_1, z_1, x_2, y_2, z_2$  are obtained using 2-adic arguments made possible by assuming  $c$  is even (using Scott and Styer 2016).

Using deep methods, many useful theoretical results are obtained, later used and improved to obtain useful general results, especially in Section 4 of Miyazaki and Pink 2025 (see especially Lemmas 4.8 and 4.9, relevant in attacking the *atmost1* problem).

The extensive computer calculations (Miyazaki and Pink, I (2024)) use methods they later refine in attacking the *atmost1* problem.

*atmost2* for  $a^x + b^y = c^z$  with  $\gcd(a, b) > 1$

For the case  $\gcd(a, b) > 1$ , we find four infinite families of  $(a, b, c)$  giving two solutions to the equation

$$a^x + b^y = c^z, a > 1, b > 1, c > 1, \gcd(a, b) > 1. \quad (1)$$

These infinite families are

(i).  $(a, b, c) = (2, 2^u(2^{h-1} - 1), 2^u(2^{h-1} + 1))$ ,  $(x_1, y_1, z_1) = (u + 1, 1, 1)$  and  $(x_2, y_2, z_2) = (2u + h + 1, 2, 2)$ ,  $u > 0, h > 1$ .

(ii).  $(a, b, c) = (2 \cdot 3^t, 3, 3)$ ,  $(x_1, y_1, z_1) = (1, t, t + 1)$  and  $(x_2, y_2, z_2) = (3, 3t, 3t + 2)$ ,  $t > 0$ .

(iii).  $(a, b, c) = (g^j, g^{ju}d, g^{ju}(d + 1))$ ,  $(x_1, y_1, z_1) = (u, 1, 1)$  and  $(x_2, y_2, z_2) = (ku + \frac{w}{j}, k, k)$  where  $j > 0, u > 0, d > 0, 2 \nmid g > 1$ ,  $(d + 1)^k - d^k = g^w$ ,  $w > 0, k > 1$ .

(iv).  $(a, b, c) = (2^i g^j, 2^{iu-1} g^{ju}d, 2^{iu-1} g^{ju}(d + 2))$ ,  $(x_1, y_1, z_1) = (u, 1, 1)$  and  $(x_2, y_2, z_2) = (ku + \frac{w}{j}, k, k)$ , where  $i > 0, j > 0, u > 0, d > 0, 2 \nmid d, 2 \nmid g > 1$ ,  $g^w$  is the greatest odd divisor of  $(d + 2)^k - d^k$ ,  $w > 0, 2 \mid k > 0$ , and  $k - v = h - (iw/j)$ , where  $2^h \parallel 2d + 2$  and  $2^v \parallel k$ .

The infinite family (i) is a subset of the infinite family (iv), but for the purposes of clarity in the proofs, we keep them distinct by taking  $g > 1$  in (iv).

We find ten further cases of  $(a, b, c)$  (with  $\gcd(a, b) > 1$ ) allowing two solutions  $(x, y, z)$  which are not members of any of the four infinite families, and which we refer to as *anomalous solutions*:

$$(a, b, c) = (2, 6, 38), (x_1, y_1, z_1) = (1, 2, 1) \text{ and } (x_2, y_2, z_2) = (5, 1, 1).$$

$$(a, b, c) = (3, 6, 15), (x_1, y_1, z_1) = (2, 1, 1) \text{ and } (x_2, y_2, z_2) = (2, 3, 2).$$

$$(a, b, c) = (6, 15, 231), (x_1, y_1, z_1) = (1, 2, 1) \text{ and } (x_2, y_2, z_2) = (3, 1, 1).$$

$$(a, b, c) = (3, 1215, 6), (x_1, y_1, z_1) = (4, 1, 4) \text{ and } (x_2, y_2, z_2) = (8, 1, 5).$$

$$(a, b, c) = (3, 6, 7857), (x_1, y_1, z_1) = (4, 5, 1) \text{ and } (x_2, y_2, z_2) = (8, 4, 1).$$

$$(a, b, c) = (5, 275, 280), (x_1, y_1, z_1) = (1, 1, 1) \text{ and } (x_2, y_2, z_2) = (7, 1, 2).$$

$$(a, b, c) = (5, 280, 78405), (x_1, y_1, z_1) = (1, 2, 1) \text{ and } (x_2, y_2, z_2) = (7, 1, 1).$$

$$(a, b, c) = (30, 70, 4930), (x_1, y_1, z_1) = (1, 2, 1) \text{ and } (x_2, y_2, z_2) = (5, 2, 2).$$

$$(a, b, c) = (30, 4930, 24304930), (x_1, y_1, z_1) = (1, 2, 1) \text{ and}$$

$$(x_2, y_2, z_2) = (5, 1, 1).$$

$$(a, b, c) = (2, 88, 6), (x_1, y_1, z_1) = (7, 1, 3) \text{ and } (x_2, y_2, z_2) = (5, 2, 5).$$

# Results for $atmost2$ , $\gcd(a, b) > 1$

We consider solutions  $(x, y, z)$  to the equation

$$a^x + b^y = c^z \quad (1)$$

for integers  $a, b, c$  all greater than one with a common divisor greater than one. Let  $Q$  be the set of all primes  $p \mid \gcd(a, b, c)$ . For every prime  $p \in Q$  let  $p^{\alpha_p} \parallel a$ ,  $p^{\beta_p} \parallel b$ ,  $p^{\gamma_p} \parallel c$ . Let  $a_1$  be the greatest divisor of  $a$  not divisible by any prime in  $Q$ , and define  $b_1$  and  $c_1$  similarly for  $b$  and  $c$ .

## Lemma (1)

*Let  $(x, y, z)$  be a solution to (1). Then, for  $p \in Q$ , two members of the triple  $\{\alpha_p x, \beta_p y, \gamma_p z\}$  must be equal and these two members must be less than or equal to the third member.*

A solution to (1) is Type A for  $p$  (where  $p \in Q$ ) if  $\alpha_p x > \beta_p y = \gamma_p z$ ; a solution to (1) is Type B for  $p$  if  $\beta_p y > \alpha_p x = \gamma_p z$ ; a solution to (1) is Type C for  $p$  if  $\gamma_p z > \alpha_p x = \beta_p y$ ; a solution to (1) is Type O for  $p$  if  $\alpha_p x = \beta_p y = \gamma_p z$ .

## Lemma (2)

*If (1) has a solution which is Type A for some prime  $p \in Q$ , and if this solution is not Type A for some prime  $q \in Q$ , then (1) can have no solutions which are Type B, C, or O for  $p$ .*

## Proof.

If a solution  $(x, y, z)$  to (1.1) is Type A for some  $p \in Q$ , and is not Type A for some prime  $q \in Q$ , then  $\alpha_p x > \beta_p y$ ,  $\alpha_p x > \gamma_p z$ ,  $\alpha_q x \leq \beta_q y$ , and  $\alpha_q x \leq \gamma_q z$ . So  $\frac{\alpha_p}{\beta_p} > \frac{\alpha_q}{\beta_q}$  and  $\frac{\alpha_p}{\gamma_p} > \frac{\alpha_q}{\gamma_q}$ . If (1) has a solution  $(x, y, z)$  which is Type B, C or O for  $p$  then  $\alpha_p x \leq \beta_p y$  and  $\alpha_p x \leq \gamma_p z$ , so that  $\alpha_q x < \beta_q y$  and  $\alpha_q x < \gamma_q z$ , contradicting the previous Lemma. □

# Solution of Type A for a given prime $p \in Q$

$y\beta_p = z\gamma_p$  so there exists a positive integer  $n = n(x, y, z)$  such that

$$y = ns, z = nt$$

where  $\beta_p/\gamma_p = t/s$ ,  $\gcd(s, t) = 1$ . For a specific solution  $(x_i, y_i, z_i)$  we write  $n(x_i, y_i, z_i) = n_i$ .

$A = \{q \in Q : \frac{\beta_q}{\gamma_q} = \frac{t}{s}\}$ .  $B = \{q \in Q : \frac{\beta_q}{\gamma_q} > \frac{t}{s}\}$ .  $C = \{q \in Q : \frac{\beta_q}{\gamma_q} < \frac{t}{s}\}$ .  
 $q \in B$  implies  $(x, y, z)$  is Type B,  $q \in C$  implies  $(x, y, z)$  is Type C,  $q \in A$  implies  $(x, y, z)$  is either Type A or Type O.

$$a_1^x \prod_{q \in Q} q^{\alpha_q x} + b_1^{ns} \prod_{q \in Q} q^{\beta_q ns} = c_1^{nt} \prod_{q \in Q} q^{\gamma_q nt}.$$

Divide by

$$D = \prod_{q \in Q} (q^{\min(s\beta_q, t\gamma_q)})^n.$$

Define  $f(n) = R^n - S^n$

$$R^n - S^n = (c_1^t \prod_{q \in C} q^{\gamma_q t - \beta_q s})^n - (b_1^s \prod_{q \in B} q^{\beta_q s - \gamma_q t})^n = a_1^x \prod_{q \in A} q^{\alpha_q x - \gamma_q n t} = f(n)$$

We assume there are no Type O solutions. Now assume there are two solutions  $(x_1, y_1, z_1)$  and  $(x_2, y_2, z_2)$ . Then  $P(f(n_1)) = P(f(n_2))$  and a standard elementary lemma gives

$$R + S = 2^h, h \geq 2; n_1 = 1, n_2 = 2; f(n_1) = 2j, 2 \nmid j; f(n_2) = 2^{h+1}j.$$

This leads to three cases, each with exactly two solutions of Type A for  $p$ :

$$(a_1, b_1, c_1) = (1, 2^{h-1} - 1, 2^{h-1} + 1), Q = \{2\}, h > 2, \quad (3)$$

$$(a_1, b_1, c_1) = (1, 7, 3), Q = \{2\}, \quad (4)$$

$$(a_1, b_1, c_1) = (1, 1, 3), Q = \{2\}. \quad (5)$$

Easy to show no third solution of Type C: recall  $\frac{\beta_p}{\gamma_p} = \frac{t}{s}$ ,  $s = t = 1$  for this case.

## Also no solution of Type B for this case

In considering the possibility of a third solution for this case, the only difficulty is eliminating Type B for case (3). We have

$$2^{z\beta} + 2^{y\beta} b_1^y = 2^{z\beta} (b_1 + 2)^z$$

$$2^{(y-z)\beta} b_1^y = (b_1 + 2)^z - 1, y > z, (y - z)\beta \geq h - 1.$$

$$(h-1)\log(2) + z\log(b_1) < z\log(b_1+2) = z\log(b_1(1+\frac{2}{b_1})) < z\log(b_1) + \frac{2z}{b_1}.$$

$z$  is the only value of  $u$  such that  $\text{rad}((b_1 + 2)^u - 1) = \text{rad}(2b_1)$ .  $z$  is the least  $u$  such that  $b_1 \mid (b_1 + 2)^u - 1$ , so that  $z \leq \varphi(b_1) \leq b_1 - 1$ .

$$(h-1)\log(2) < \frac{2(b_1-1)}{b_1} < 2, h=3$$

giving  $2^{(y-z)\beta} 3^y = 5^z - 1$ , impossible. (Cases (4) and (5) are easier.)

**Conclusion:** two Type A solutions for  $p$  with no Type O for  $q \in Q$  implies exactly two solutions  $(x, y, z)$ .

# Preliminary Result for Type C

## Proposition (1)

*Assume (1) has a solution which is Type C for some prime  $p \in Q$ . Then (1) can have no solution which is Type A, B, or O for  $p$ .*

Proof:

Let  $p$  be any prime in  $Q$ . For brevity, write  $\alpha$  for  $\alpha_p$ ,  $\beta$  for  $\beta_p$ ,  $\gamma$  for  $\gamma_p$ . Let  $(x_c, y_c, z_c)$  be a solution to (1) which is Type C for  $p$  and let  $(x_d, y_d, z_d)$  be a solution to (1) which is Type A, B, or O for  $p$ . By Lemma 1  $\alpha x_c < \gamma z_c$  and  $\alpha x_d \geq \gamma z_d$  so that

$$\frac{x_c}{z_c} < \frac{x_d}{z_d}. \quad (6)$$

Now let  $a_p = a/p^\alpha$ ,  $b_p = b/p^\beta$  and let  $c_p = c/p^\gamma$ .  $a^{x_d} < c^{z_d}$  so  $p^{\alpha x_d} a_p^{x_d} < p^{\gamma z_d} c_p^{z_d}$ , so that  $p^{\alpha x_d - \gamma z_d} a_p^{x_d} < c_p^{z_d}$ , so that

$$x_d \log(a_p) < z_d \log(c_p). \quad (7)$$

We can take  $a^{x_c} \geq b^{y_c}$ . Then  $2a^{x_c} \geq c^{z_c}$ , so that  $2a_p^{x_c} \geq p^{\gamma z_c - \alpha x_c} c_p^{z_c} \geq 2c_p^{z_c}$ , so that

$$x_c \log(a_p) \geq z_c \log(c_p). \quad (8)$$

Since (7) requires  $c_p > 1$ , (8) gives  $a_p > 1$ . So (8) in combination with (7) gives

$$\frac{x_c}{z_c} > \frac{x_d}{z_d},$$

contradicting (6). So the existence of a solution of Type C for  $p$  makes solutions of Type A, B, or O for  $p$  impossible.

# Preliminary Result for Type O

## Proposition (2)

*For a given  $p \in Q$ , there is at most one solution  $(x, y, z)$  to (1) which is Type O for  $p$ .*

## Proof.

Let  $r, s$ , and  $t$  be positive integers with  $\gcd(r, s, t) = 1$  such that

$$r\alpha_p = s\beta_p = t\gamma_p, p \in Q.$$

By Lemma (1) any solution  $(x, y, z)$  which is Type O for  $p$  must have

$$x = nr, y = ns, z = nt,$$

for some positive integer  $n$ . By Fermat's Last Theorem,  $n \leq 2$ . So if there are two distinct solutions  $(x, y, z)$  to (1) which are Type O for  $p$ , then  $a^r + b^s = c^t$  and  $a^{2r} + b^{2s} = c^{2t}$ , which is impossible since  $c^{2t} = (a^r + b^s)^2 > a^{2r} + b^{2s} = c^{2t}$ . □

## Solution of Type C for a given prime $p \in Q$

Let  $S_c$  be the set of all triples  $(a, b, c)$  such that  $\gcd(a, b) > 1$ ,  $N(a, b, c) > 1$ ,  $a$ ,  $b$ , and  $c$  are not all powers of 2, and there exists a solution to (1) which is Type C for some prime in  $Q$ .

### Proposition (3)

*For  $(a, b, c) \in S_c$ , (1) has exactly two solutions  $(x_1, y_1, z_1)$  and  $(x_2, y_2, z_2)$ ,  $N(a, b, c) = 2$ .*

Proof:

Using the same basic steps as used for Type A:

- establish  $f(n) = R^n + S^n$ .
- eliminate Type O solutions to obtain  $P(f(n_1)) = P(f(n_2))$ .
- obtain exactly two solutions of Type C for  $p$ :

$$R = 2, S = 1, n_1 = 1, n_2 = 3; f(n_1) = 3, f(n_2) = 9$$

Using Preliminary Result on Type C and Type O above, we show there is no third solution of Type A or B for  $p$ .

# Handling Type O

Let  $(x_1, y_1, z_1)$  be a solution to (1) which is Type O for some prime  $p \in Q$ , with a second solution  $(x_2, y_2, z_2)$ . Using the Preliminary Result on Types C and O along with Lemma (2), we find we can take  $(x_2, y_2, z_2)$  to be Type A for every prime  $p \in Q$ . Letting  $\frac{\beta_p}{\gamma_p} = \frac{t}{s}$ ,  $\gcd(s, t) = 1$ , we can take

$y_1 = n_1 s, z_1 = n_1 t, y_2 = n_2 s, z_2 = n_2 t, n_1 \in \mathbb{Z}^+, n_2 \in \mathbb{Z}^+, n_1 \neq n_2$ , we easily show that  $(x_1, y_1, z_1)$  must be either Type O or Type A for every prime in  $Q$ . Let  $G = \{q \in Q : (x_1, y_1, z_1) \text{ Type O for } q\}$ ,  $H = \{q \in Q : (x_1, y_1, z_1) \text{ Type A for } q\}$ . Then, for  $g$  such that  $P(g) = G$ ,

$$a = a_1 g^{\alpha_g} \prod_{q \in H} q^{\alpha_q}, b = b_1 g^{\beta_g} \prod_{q \in H} q^{\beta_q}, c = c_1 g^{\gamma_g} \prod_{q \in H} q^{\gamma_q}$$

$$f(n_1) = c_1^{n_1 t} - b_1^{n_1 s} = a_1^{x_1} \prod_{q \in H} q^{\alpha_q x_1 - n_1 t \gamma_q}, x_1 = \frac{n_1 t \gamma_g}{\alpha_g} \quad (9)$$

$$f(n_2) = c_1^{n_2 t} - b_1^{n_2 s} = a_1^{x_2} g^{\alpha_g x_2 - n_2 t \gamma_g} \prod_{q \in H} q^{\alpha_q x_2 - n_2 t \gamma_q}, x_2 > \frac{n_2 t \gamma_g}{\alpha_g}. \quad (10)$$

From (9) and (10) we derive

$$\frac{x_2}{x_1} > \frac{n_2}{n_1}. \quad (11)$$

Let  $U$  be the product of all primes in  $H$ , let  $r$  be any prime dividing  $a_1 U$  (if such  $r$  exists), and let  $r^{v_1} \parallel f(n_1)$  and  $r^{v_2} \parallel f(n_2)$ . Recalling (9) and (10) we see that, if  $r \mid a_1$ , then  $\frac{v_2}{v_1} = \frac{x_2}{x_1} > \frac{n_2}{n_1}$  by (11). If  $r \in H$  then by (9), (10), and (11)

$$\frac{v_2}{v_1} = \frac{\alpha_r x_2 - n_2 t \gamma_r}{\alpha_r x_1 - n_1 t \gamma_r} = \frac{\alpha_r \frac{x_2}{x_1} x_1 - \frac{n_2}{n_1} n_1 t \gamma_r}{\alpha_r x_1 - n_1 t \gamma_r} > \frac{n_2}{n_1}.$$

So in either case, we have

$$\frac{v_2}{v_1} > \frac{n_2}{n_1}. \quad (12)$$

With a few elementary calculations we then show  $n_1 \mid n_2$ . If  $a_1 U > 1$  and  $r^{v_1} > 2$ , then

$$r^{v_2 - v_1} \mid \frac{n_2}{n_1}, \frac{n_2}{n_1} \geq r^{v_2 - v_1} \geq r^{\frac{v_2}{v_1} - 1} > r^{\frac{n_2}{n_1} - 1}.$$

So we must have  $r^{v_1} \leq 2$ , which requires one of these cases:

Case 1:  $a_1 = 1$ ,  $H = \{2\}$ ,  $f(n_1) = 2$ .

Case 2:  $a_1 = 2$ ,  $H$  is empty,  $f(n_1) = 2$ .

Case 3:  $a_1 = 1$ ,  $H$  is empty,  $f(n_1) = 1$ .

For all three cases,  $n_1 = 1$ .

We handle each of these cases using an intricate elementary argument involving special properties of infinite families to obtain

### Proposition (4)

*Let  $S_o$  be the set of all triples  $(a, b, c)$  such that  $\gcd(a, b) > 1$ ,  $N(a, b, c) > 1$ , and there exists a solution to (1) which is Type O for some prime in  $Q$ . For  $(a, b, c) \in S_o$  we have  $N(a, b, c) = 2$ , and (1) has two solutions  $(x_1, y_1, z_1)$  and  $(x_2, y_2, z_2)$ .*

Let  $S_{aa}$  be the set of all triples  $(a, b, c)$  such that  $\gcd(a, b) > 1$  and (1) has two distinct solutions  $(x, y, z)$  both of which are Type A for some prime in  $Q$ . Define  $S_{bb}$  similarly. Using Proposition (4) and the previous **Conclusion** concerning Type A solutions, we obtain:

### Proposition (5)

*For  $(a, b, c) \in S_{aa}$  we have  $N(a, b, c) = 2$ , and (1) has two solutions  $(x_1, y_1, z_1)$  and  $(x_2, y_2, z_2)$ . Similarly for  $S_{bb}$ .*

From Propositions (3), (4), and (5) we obtain

### Theorem

*Let  $S_i = S_c \cup S_o \cup S_{aa} \cup S_{bb}$ . If  $(a, b, c)$  in  $S_i$ , then  $N(a, b, c) = 2$ , and (1) has two solutions  $(x_1, y_1, z_1)$  and  $(x_2, y_2, z_2)$ .*

If  $(a, b, c) \notin S_i$  then (1) has no solutions of Type C or O for any prime in  $Q$ , and at most one solution of Type A and at most one solution of Type B. Let  $N(a, b, c)$  be the number of solutions to (1) where two solutions  $(x_1, y_1, z_1)$  and  $(x_2, y_2, z_2)$  are considered the same solution if  $\{a_1^x, b_1^y\} = \{a^{x_2}, b^{y_2}\}$ . So we obtain our main theorem for  $\gcd(a, b) > 1$ :

### Theorem

*If  $a, b, c$  are positive integers all greater than one with  $\gcd(a, b) > 1$  and with at least one of  $a, b, c$  not a power of 2, then  $N(a, b, c) \leq 2$ .*

If we redefine  $N(a, b, c)$  so that two solutions are considered the same solution if  $a = b$  and  $\{x_1, y_1\} = \{x_2, y_2\}$ , we obtain the following cases with three solutions:  $\{a, b\} = \{2, 8\}$ ,  $c = 2^{3t} \cdot 3$ , but since this solution is a member of the infinite family (i), we use the original definition of  $N(a, b, c)$ .

## Towards *atmost1*, $\gcd(a, b) > 1$

If  $a^x + b^y = c^z$  with  $\gcd(a, b) > 1$  has more than one solution and is not in any of the four infinite families, then

- $a = g^{\alpha_g} a_1, b = g^{\beta_g} b_1, c = g^{\gamma_g} c_1, \gcd(g, a_1 b_1 c_1) = 1, \gcd(a_1, b_1) = 1,$
- exactly two solutions, one of which is Type A for  $g$ , and the other of which is Type B for  $g$ ,
- $\{a^{x_1}, b^{y_1}\} \neq \{a^{x_2}, b^{y_2}\}.$

From this we derive two equations with relatively prime terms

$$g^{w_1} a_1^{x_1} + b_1^{y_1} = c_1^{z_1}, a_1^{x_2} + g^{w_2} b_1^{y_2} = c_1^{z_2}.$$

For any pair of equations with relatively prime terms which can be represented as in the above display, we consider whether there exist  $\alpha, \beta$ , and  $\gamma$  which produce an  $(a, b, c) \in S_j$ . To do this we construct systems of linear equations in which the unknowns are  $\alpha, \beta$ , and  $\gamma$ :

$$y_1 \beta = z_1 \gamma, y_2 \beta - w_2 = z_2 \gamma,$$

$$x_1 \alpha - w_1 = z_1 \gamma, x_2 \alpha = z_2 \gamma.$$

## Towards *atmost1*, $\gcd(a, b) = 1$ . Conjecture (2016)

$$N(3, 5, 2) = 3 : (x, y, z) = (1, 1, 3), (3, 1, 5), (1, 3, 7),$$

$$N(3, 13, 2) = 2 : (x, y, z) = (1, 1, 4), (5, 1, 8),$$

$$N(3, 10, 13) = 2 : (x, y, z) = (1, 1, 1), (7, 1, 3),$$

$$N(2, 89, 91) = 2 : (x, y, z) = (1, 1, 1), (13, 1, 2),$$

$$N(2, 3, 11) = 2 : (x, y, z) = (1, 2, 1), (3, 1, 1),$$

$$N(2, 3, 35) = 2 : (x, y, z) = (5, 1, 1), (3, 3, 1),$$

$$N(2, 3, 259) = 2 : (x, y, z) = (8, 1, 1), (4, 5, 1),$$

$$N(2, 5, 133) = 2 : (x, y, z) = (7, 1, 1), (3, 3, 1),$$

$$N(2, 91, 8283) = 2 : (x, y, z) = (13, 1, 1), (1, 2, 1),$$

$$N(3, 13, 2200) = 2 : (x, y, z) = (7, 1, 1), (1, 3, 1),$$

$$N(2, 5, 3) = 2 : (x, y, z) = (1, 2, 3), (2, 1, 2),$$

$$N(2, 3, 5) = 2 : (x, y, z) = (1, 1, 1), (4, 2, 2),$$

$$N(2, 7, 3) = 2 : (x, y, z) = (1, 1, 2), (5, 2, 4),$$

$$N(2, 2^r - 1, 2^r + 1) = 2 : (x, y, z) = (1, 1, 1), (r + 2, 2, 2), r > 3.$$

## Towards *atmost1Pillai*, $\gcd(a, b) = 1$

For the Pillai case, we use notation to match the more general case:

$$c^z - a^x = b, \gcd(a, c) = 1. \quad (1)$$

### Theorem (Miyazaki and Pink (2025))

*For any fixed  $c$ , there is at most one solution to (1) except for only finitely many pairs of  $a$  and  $b$ .*

Later, an effective result for a special set of  $c$  was given.

### Theorem (Miyazaki, Scott, Styer)

*Let  $c$  be any fixed prime of the form  $2^r \cdot 3 + 1$  with some positive integer  $r$  such that  $r \leq 3912$ , namely,*  
 $r \in \{1, 2, 5, 6, 8, 12, 18, 30, 36, 41, 66, 189, 201, 209, 276, 353, 408, 438, 534, 2208, 2816, 3168, 3189, 3912\}.$

*Then there is at most one solution to equation (1) except for  $(a, b, c) = (3, 10, 13)$ , where there are exactly two solutions.*

# Towards *atmost1* $a^x + b^y = c^z$ , $\gcd(a, b) = 1$

In their second paper, Miyazaki and Pink also deal with:

$$a^x + b^y = c^z, \tag{2}$$

**Definition:** Let  $M$  be a positive integer. For any integer  $A$  coprime to  $M$ , the extended multiplicative order of  $A$  modulo  $M$  is defined as the least positive integer  $E$  such that  $A^E$  is congruent to 1 or -1 modulo  $M$ .

## Theorem

*Assume that the extended multiplicative orders of  $a$  and  $b$  modulo  $c$  are relatively prime. Then the above Conjecture is true, namely, there is at most one solution to equation (2), except when  $(a, b, c)$  or  $(b, a, c)$  equals one of  $(3, 5, 2)$ ,  $(3, 13, 2)$ ,  $(2, 5, 3)$  and  $(2, 7, 3)$ .*

## Corollary

*Assume that at least one of  $a$  and  $b$  is congruent to 1 or -1 modulo  $c$ . Then there is at most one solution to (2), except when  $(a, b, c)$  or  $(b, a, c)$  equals one of  $(3, 5, 2)$ ,  $(3, 13, 2)$ ,  $(2, 5, 3)$  and  $(2, 7, 3)$ .*

In the same paper, they also prove

### Theorem

*For any fixed  $c$  satisfying  $\max\{2^{v_2(c)}, 3^{v_3(c)}\} > \sqrt{c}$ , there is at most one solution to (2) except for only finitely many pairs  $(a, b)$ .*

### Theorem

*If  $c \in \{5, 17, 257, 65537\}$ , then the above conjecture is true, namely, there is at most one solution to (2), except when  $(a, b)$  or  $(b, a)$  is  $(c - 2, 2)$ .*

These two theorems, combined with later work on the case  $c = 2^r \cdot 3 + 1$ , handles many values of  $c$  as we will show shortly. A related result is

### Theorem

*The above conjecture is true for infinitely many values of  $c$  which are not perfect powers.*

# Primes of form $2^r \cdot 3 + 1$

## Theorem (Miyazaki and Pink (2025))

*If  $c$  is a prime of the form  $2^r \cdot 3 + 1$  with some positive integer  $r$ , then  $N(a, b, c) \leq 1$  except for finitely many pairs  $(a, b)$ .*

The above theorem is ineffective. An effective version was later given by Miyazaki, Scott, Styer: all pairs  $(a, b)$  for a given  $c$  can be found effectively. Previously,  $c = 13 = 2^2 \cdot 3 + 1$  was handled completely (Miyazaki and Pink 2025), and later  $c = 7 = 2 \cdot 3 + 1$  and  $c = 97 = 2^5 \cdot 3 + 1$  were handled completely (Miyazaki and Styer). This combined with the results on the previous slide and earlier results on  $c = 2$  (Scott, 1993) allows many values of  $c$  to be handled, either completely or with a finite amount of effectively determined calculation:

2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20  
21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40  
41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60  
61, 62, 63, 64, 65, 66, 67, 68, 69, 70, 71, 72, 73, 74, 75, 76, 77, 78, 79, 80  
81, 82, 83, 84, 85, 86, 87, 88, 89, 90, 91, 92, 93, 94, 95, 96, 97, 98, 99  
100, 101, 102, 103, 104, 105, 106, 107, 108, 109, 110  
111, 112, 113, 114, 115, 116, 117, 118, 119, 120  
121, 122, 123, 124, 125, 126, 127, 128, 129, 130  
131, 132, 133, 134, 135, 136, 137, 138, 139, 140

Values of  $c$  handled completely

For given  $c$ ,  $(a, b)$  effectively computable

# Reporting on $a, b \leq 20$

## Theorem

*There are exactly 53 solutions  $(x, y, z, c)$  in positive integers to  $a^x + b^y = c^z$ , where  $2 \leq a < b \leq 20$ ,  $a$  and  $b$  not perfect powers,  $z \geq 2$ , and  $c \geq 2$ . The solutions all have  $x \leq 9$ ,  $y \leq 4$ .*

## Theorem (Cazorla García (2026))

*Let  $C_1, q$  be integers with  $1 \leq C_1 \leq 20$  and  $2 \leq q < 25$ , with  $C_1$  squarefree and  $q$  prime. Then, all integer solutions  $(x, y, \alpha, n)$  to the equation  $C_1 x^2 + q^\alpha = y^n$ ,  $\gcd(C_1 x, q, y) = 1$ ,  $x, y > 0$ ,  $\alpha > 0$ ,  $n \geq 3$ , are listed in Table 1 of the paper.*

This theorem takes care of almost all cases of  $a^x + b^y = c^z$  for  $z \geq 3$ . For  $z = 2$ , the paper by Bennett and Billerey (2016) along with elementary methods cover all but four cases:  $5^x + 11^y = c^2$ ,  $5^x + 19^y = c^2$ ,  $11^x + 14^y = c^2$ ,  $17^x + 19^y = c^2$ . Then apply the ideas of Bennett-Billerey to the  $S$ -unit sets  $\{5, 11\}$ ,  $\{5, 19\}$ ,  $\{2, 7, 11\}$ ,  $\{17, 19\}$ .

# Further Generalizations

Miyazaki and Pink (2025) used the following lemma concerning two solutions  $(x_1, y_1, z_1)$  and  $(x_2, y_2, z_2)$  to  $a^x + b^y = c^z$ :

## Lemma

*Under the abc conjecture, if  $\max\{a, b\}$  exceeds some constant depending only on  $c$ , then  $(x_1, x_2, y_1, y_2) = (2, 1, 1, 2)$  or  $(1, 2, 2, 1)$ .*

to obtain the equivalent of the following:

## Theorem

*Assuming the truth of the abc conjecture, for a given  $c$  there are only a finite number of pairs  $(a, b)$  for which  $a^x + b^y = c^z$  gives more than one solution, provided  $c$  is not a composite number for which  $3 \mid \phi(c)$ , where  $\phi(n)$  is the totient function of  $n$ .*

Later Le and Miyazaki (2026) obtained an improvement:

Le and Miyazaki (2026) removed the dependence on the *abc* conjecture by using a theorem of Ridout. They also shortened the treatment by using a nice elementary proof to show that the only case of  $(x_1, y_1, x_2, y_2) = (1, 2, 2, 1)$  or  $(2, 1, 1, 2)$  is given by  $(a, b, c, x_1, y_1, z_1, x_2, y_2, z_2) = (2, 5, 3, 1, 2, 3, 2, 1, 2)$ .

### Theorem (Le and Miyazaki (2026))

*For given  $c$ , there are only a finite number of pairs  $(a, b)$  for which  $a^x + b^y = c^z$  gives more than one solution.*

# The unhandled case, prime bases

The question of two solutions to  $a^x + b^y = c^z$  for  $a, b, c$  all prime has been narrowed to consideration of one particular case (Le, Scott, Styer (2024)): If two solutions  $(x_1, y_1, z_1)$  and  $(x_2, y_2, z_2)$  to  $a^x + b^y = c^z$  occur and  $(a, b, c)$  is not equal to  $(2, 3, 5)$ ,  $(2, 3, 11)$ ,  $(2, 5, 3)$ ,  $(2, 7, 3)$ ,  $(3, 5, 2)$ ,  $(3, 13, 2)$ , we must have

$$2^{x_1} + b^{y_1} = c$$

and

$$2^{x_2} + b^{y_2} = c^{z_2}, z_2 > 1.$$

Red exponents are even, blue exponents are odd.

From these two equations the following restrictions are derived:

# The unhandled case, continued

- $a = 2$ ,  $b \equiv 1 \pmod{48}$ ,  $c \equiv 17 \pmod{48}$ ;
- $b > 10^9$ ,  $c > 10^{18}$ ;
- at least one of the multiplicative orders  $u_b(c)$  or  $u_c(b)$  must be odd (where  $u_p(n)$  is the least integer  $t$  such that  $n^t \equiv 1 \pmod{p}$ );
- 2 must be an octic residue modulo  $c$  except for one specific case;
- $2 \mid v_2(b-1) \leq v_2(c-1)$  (where  $v_2(n)$  satisfies  $2^{v_2(n)} \parallel n$ );
- there must be exactly two solutions  $(x_1, y_1, z_1)$  and  $(x_2, y_2, z_2)$  with  $1 = z_1 < z_2$  and either  $x_1 \geq 28$  or  $x_2 \geq 88$ .

## Unlikelihood of further solutions

$$Q = \frac{\log(c)}{\log(\text{rad}(abc))}$$

Then for the equation  $2^{x_2} + b^{y_2} = c^{z_2}$  we have

$$\begin{aligned} Q &= \frac{z_2 \log(c)}{\log(2) + \log(b) + \log(c)} \geq \frac{3 \log(c)}{(3/2) \log(c) + \log(2)} \\ &= 2 - \frac{2 \log(2)}{(3/2) \log(c) + \log(2)} > 1.97. \end{aligned}$$

The highest value for  $Q$  found in recent researches on the *abc* conjecture is  $Q = 1.62991$  for  $(a, b, c) = (2, 3^{10} \cdot 109, 23^5)$ . If  $z_2 > 3$ , then we have  $Q > 3.29$ : if a conjecture of Tenenbaum (quoted in Section B19 of Guy) is true, then  $Q = 3.29$  is impossible, so that  $z_2 = 3$ .

# Characterizing Cases with $N(a, b, c) = 2$

In this section we treat the cases  $\gcd(a, b) = 1$  and  $\gcd(a, b) > 1$  together. For both  $\gcd=1$  and  $\gcd>1$  the treatment for *atmost2* is complete, giving the following theorem:

## Theorem

*If  $a, b, c$  are positive integers all greater than one with at least one of  $a, b, c$  not a power of 2 and with  $(\{a, b\}, c) \neq (\{3, 5\}, 2)$ , then  $N(a, b, c) \leq 2$ .*

If we eliminate from consideration the two exceptional cases in the above theorem, then the following is a complete list of all known solutions which are not members of known infinite families:

$$2^2 + 5 = 3^2, 2 + 5^2 = 3^3$$

$$2 + 7 = 3^2, 2^5 + 7^2 = 3^4$$

$$2^3 + 3 = 11, 2 + 3^2 = 11$$

$$2^5 + 3 = 35, 2^3 + 3^3 = 35$$

$$2^7 + 5 = 133, 2^3 + 5^3 = 133$$

$$3 + 13 = 2^4, 3^5 + 13 = 2^8$$

$$2^8 + 3 = 259, 2^4 + 3^5 = 259$$

$$3 + 10 = 13, 3^7 + 10 = 13^3$$

$$3^7 + 13 = 2200, 3 + 13^3 = 2200$$

$$2 + 89 = 91, 2^{13} + 89 = 91^2$$

$$2^{13} + 91 = 8283, 2 + 91^2 = 8283$$

$$2 + 6^2 = 38, 2^5 + 6 = 38$$

$$3^2 + 6 = 15, 3^2 + 6^3 = 15^2$$

$$6 + 15^2 = 231, 6^3 + 15 = 231$$

$$3^4 + 1215 = 6^4, 3^8 + 1215 = 6^5$$

$$3^4 + 6^5 = 7857, 3^8 + 6^4 = 7857$$

$$5 + 275 = 280, 5^7 + 275 = 280^2$$

$$5 + 280^2 = 78405, 5^7 + 280 = 78405$$

$$30 + 70^2 = 4930, 30^5 + 70^2 = 4930^2$$

$$30 + 4930^2 = 24304930,$$

$$30^5 + 4930 = 24304930$$

$$2^7 + 88 = 6^3, 2^5 + 88^2 = 6^5$$

Each member of this list, except for the two members in red, satisfies either  $y_1 = y_2$  or  $z_1 = z_2$ , so that every member of this list of 21 anomalous solutions is either on Bennett's original list of Pillai cases or is a rearrangement of one of the members on Bennett's Pillai list.

So, to characterize cases with exactly two solutions, we re-examine the original list of Pillai cases given by Bennett for  $a^x + b = c^z$ :

$$3 - 2 = 3^2 - 2^3 = 1$$

$$2^3 - 3 = 2^5 - 3^3 = 5$$

$$2^4 - 3 = 2^8 - 3^5 = 13$$

$$2^3 - 5 = 2^7 - 5^3 = 3$$

$$13 - 3 = 13^3 - 3^7 = 10$$

$$91 - 2 = 91^2 - 2^{13} = 89$$

$$6 - 2 = 6^2 - 2^5 = 4$$

$$15 - 6 = 15^2 - 6^3 = 9$$

$$280 - 5 = 280^2 - 5^7 = 275$$

$$4930 - 30 = 4930^2 - 30^5 = 4900$$

$$6^4 - 3^4 = 6^5 - 3^8 = 1215$$

All but three of these cases satisfy  $c^{z_2} - c^{z_1} = a^{x_2} - a^{x_1}$  with  $z_2 = 2z_1$  or (in the last listed case)  $x_2 = 2x_1$ . Mignotte and Petho (1999) pointed out that when  $a$  is prime, such cases correspond to Ramanujan-Nagell type equations for which all solutions are known (see Beukers (1980)). Also, the case  $(a, b, c) = (6, 9, 15)$  is handled by work of Mordell, and the case  $(a, b, c) = (3, 10, 13)$  is in the infinite family in Theorem 1 of Scott (1993).

Further, if we have  $a^2 - a = b^{y_2} - b$ , we also have two solutions to the equation

$$\pm rA^X \pm sB^Y = C, \pm \text{ independent}, r \geq 1, s \geq 1 \quad (*)$$

for the following choices of  $(A, B, C, r, s, X_1, Y_1, X_2, Y_2)$ :

$$(a, b, ab - a + b, 1, a + 1, 1, 1, 3, y_2), \quad (I)$$

$$(a - 1, b, ab - a - 2b + 1, 1, a - 2, 1, 1, 3, y_2). \quad (II)$$

If  $b = 2$ , both (I) and (II) also give the solution  $(X, Y) = (0, 0)$ , leading, in the case (II), to a Goormaghtigh equation.

Taking  $X \geq 0$ ,  $Y \geq 0$ , and  $\gcd(A, B) = 1$ , (\*) gives a general equation which has at most three solutions except for completely designated exceptional cases. The first five cases on Bennett's list of double solutions for the Pillai equation occur in infinite families giving exactly three solutions to (\*).

$$3 - 2 = 3^2 - 2^3 = 1$$

$$2^3 - 3 = 2^5 - 3^3 = 5$$

$$2^4 - 3 = 2^8 - 3^5 = 13$$

$$2^3 - 5 = 2^7 - 5^3 = 3$$

$$13 - 3 = 13^3 - 3^7 = 10$$

$$91 - 2 = 91^2 - 2^{13} = 89$$

$$6 - 2 = 6^2 - 2^5 = 4$$

$$15 - 6 = 15^2 - 6^3 = 9$$

$$280 - 5 = 280^2 - 5^7 = 275$$

$$4930 - 30 = 4930^2 - 30^5 = 4900$$

$$6^4 - 3^4 = 6^5 - 3^8 = 1215$$

$$\text{Ramanujan-Nagell } x^2 + 7 = 2^n$$

$$\text{Ramanujan-Nagell } x^2 + 11 = 3^n$$

$$\text{Ramanujan-Nagell } x^2 + 19 = 5^n$$

Derived from  $x^2 + 7$ : Type I

Derived from  $x^2 + 7$ : Type II

Infinite family from Theorem 1 of [Sc]

Mordell

Having just considered cases of two solutions to  $a^x + b = c^z$  with  $z_2 = 2$ , we now consider cases in which either  $c$  or  $a$  is equal to two, which we give in red. The cases with  $c = 2$  are known to be the only such cases (Scott (1993)). We have just shown that cases with  $a = 2$  can be derived from the Ramanujan-Nagell equation. The question of whether there are further cases with  $a = 2$  is more difficult, as pointed out by Bennett (2001).

$$3 - 2 = 3^2 - 2^3 = 1$$

$$2^3 - 3 = 2^5 - 3^3 = 5$$

$$2^4 - 3 = 2^8 - 3^5 = 13$$

$$2^3 - 5 = 2^7 - 5^3 = 3$$

$$13 - 3 = 13^3 - 3^7 = 10$$

$$91 - 2 = 91^2 - 2^{13} = 89$$

$$6 - 2 = 6^2 - 2^5 = 4$$

$$15 - 6 = 15^2 - 6^3 = 9$$

$$280 - 5 = 280^2 - 5^7 = 275$$

$$4930 - 30 = 4930^2 - 30^5 = 4900$$

$$6^4 - 3^4 = 6^5 - 3^8 = 1215$$

If we take  $c$  equal to a prime  $p$  and let  $d$  be an integer, then the equation  $p^x - 2^y = d$ , except for four known cases, has at most one solution except possibly when  $p$  is a Wieferich prime greater than  $10^{15}$  (Scott and Styer (2004)). In response to this result, Luca (2003) showed  $p^{x_1} - p^{x_2} = q^{y_1} - q^{y_2}$  has only finitely many solutions  $(p, x_1, y_1, x_2, y_2)$ ,  $p \neq q$ ,  $x_1 \neq x_2$ .