Properties of sum-of-digits functions

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Basic definition

Given an integer $q \geq 2$, every non-negative integer can be expressed in the form of the q-**ary digital expansion**

$$n = \sum_{j \geq 0} \varepsilon_j q^j,$$

where $0 \leq \varepsilon_j = \varepsilon_{q,j}(n) < q$ for every index $j \geq 0$.

The q-**ary sum-of-digits function** is defined by

$$s_q(n) = \sum_{j \geq 0} \varepsilon_j.$$

The sum is actually finite, where the q-ary digits ε_j vanish for $j > \ell = \ell(n) = \lfloor \log_q n \rfloor$.

If we rewrite the values of $s_q(n)$ in the same base q we obtain a bit unusual table

q	$s_q(n)$ for $n = 100, 101, \ldots, 115$
2	11_2, 100_2, 100_2, 101_2, 11_2, 100_2, 100_2, 101_2, 100_2, 101_2, 101_2, 110_2, 11_2, 100_2, 100_2, 101_2
3	11_3, 12_3, 11_3, 12_3, 20_3, 12_3, 20_3, 21_3, 2_3, 10_3, 11_3, 10_3, 11_3, 12_3, 11_3, 12_3
4	10_4, 11_4, 12_4, 13_4, 11_4, 12_4, 13_4, 20_4, 12_4, 13_4, 20_4, 21_4, 10_4, 11_4, 12_4, 13_4
5	4_5, 10_5, 11_5, 12_5, 13_5, 10_5, 11_5, 12_5, 13_5, 14_5, 11_5, 12_5, 13_5, 14_5, 20_5, 12_5
6	14_6, 15_6, 11_6, 12_6, 13_6, 14_6, 15_6, 20_6, 3_6, 4_6, 5_6, 10_6, 11_6, 12_6, 4_6, 5_6
7	4_7, 5_7, 6_7, 10_7, 11_7, 3_7, 4_7, 5_7, 6_7, 10_7, 11_7, 12_7, 4_7, 5_7, 6_7, 10_7
8	11_8, 12_8, 13_8, 14_8, 6_8, 7_8, 10_8, 11_8, 12_8, 13_8, 14_8, 15_8, 7_8, 10_8, 11_8, 12_8
9	4_9, 5_9, 6_9, 7_9, 8_9, 10_9, 11_9, 12_9, 4_9, 5_9, 6_9, 7_9, 8_9, 10_9, 11_9, 12_9
10	1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 2, 3, 4, 5, 6, 7
11	a_{11}, 10_{11}, 11_{11}, 12_{11}, 13_{11}, 14_{11}, 15_{11}, 16_{11}, 17_{11}, 18_{11}, a_{11}, 10_{11}, 11_{11}, 12_{11}, 13_{11}, 14_{11}
16	a_{16}, b_{16}, c_{16}, d_{16}, e_{16}, f_{16}, 10_{16}, 11_{16}, 12_{16}, 13_{16}, 14_{16}, 15_{16}, 7_{16}, 8_{16}, 9_{16}, a_{16}

Serious study of properties of the sum-of-digits functions arose in connection with divisibility problems involving factorials and binomial coefficients. They also appear in other areas of mathematics, as for instance, in:

- algebraic topology (e.g. HIRSCH)
- algebraic number theory (e.g. ORE)
- computational algorithms
- combinatorics

It is probably their applicability the reason that many of the results have been proven again and again an unusual number of times.

HIRSCH, G.: *On a property of the b-adic expression of integers*, Am. Math. Mon. **74** (1967), 561–563
ORE, Ö.: *Über den Zusammenhang zwischen den definierenden Gleichungen und der Idealtheorie in algebraischen Körpern I*, Math. Ann. **96** (1926), 313–352

$s_q(n)$ CONNECTIONS

Sum-of-digits sequences are connected to various aspects of mathematics (especially discrete one and combinatorics):

- to check the arithmetic operations of early computers
- quick divisibility tests
- Hamming weight
- Edgeworth[1] (1888) suggested using sums of 50 digits taken from mathematical tables of logarithms as a form of random number generation
- recreational mathematics
  - **sum-product number** in a given number base q is a natural number that is equal to the product of the sum of its digits and the product of its digits (in base 10 only 0, 1, 135, 144 OEIS A038369)
  - the **digital root** (also **repeated digital sum**) of a positive integer given radix q is the (single digit) value obtained by an iterative process of summing its q-digits. Digital roots are used in Western numerology. For instance, the digital root (base 10) of every even perfect number > 6 is 1.

[1] Francis Ysidro Edgeworth (1845-1926) was an Anglo-Irish philosopher and political economist who made significant contributions to the methods of statistics during the 1880s

Niven numbers

The 5th Annual Mathematics Conference at the Department of Mathematics of Miami University in 1977 was devoted to the number theory and the invited speaker was Ivan M. Niven. He mentioned a question which appeared in the children's pages of a certain newspaper: *Find a whole number which is twice the sum of its digits*. He suggested

- for professional mathematicians to find an asymptotic formula for the number of integers $n < N$ such that $s_q(n)$ divides n.

The name **Niven numbers** first appeared in an article by Kennedy *et al*. three years after his lecture.

Niven numbers are also called **harshad numbers**, a Sanskrit name (meaning *giving joy* in Sanskrit harsha, *joy*) originally defined by Indian recreational mathematician Dattatreya Ramchandra Kaprekar (1905-1986).

Kaprekar, D.R.: *Multidigital Numbers*, Scripta Math. **21** (1955), 27
Kennedy, R.E., Goodman, T., Best, C.: *Mathematical discovery and Niven numbers*, The MATYC Journal **14** (1980), 1, 20-25

Small Niven numbers

q		
2	4, 6, 8, 10, 12, 16, 18, 20, 21, 24, 32, 34, 36, 40, 42, 48	A049445
3	4, 6, 8, 9, 10, 12, 15, 16, 18, 20, 21, 24, 25, 27, 28, 30, 32	A064150
4	6, 8, 9, 12, 16, 18, 20, 21, 24, 28, 30, 32, 33, 35, 36, 40, 42	A064438
5	6, 8, 10, 12, 15, 16, 18, 20, 24, 25, 26, 27, 28, 30, 32, 36	A064481
6	10, 12, 15, 18, 20, 24, 25, 30, 36, 40, 42, 44, 45, 48, 50, 55	
7	8, 9, 12, 14, 15, 16, 18, 21, 24, 27, 28, 30, 32, 35, 36, 40, 42	
8	14, 16, 21, 24, 28, 32, 35, 40, 42, 48, 49, 56, 64, 66, 70, 72	A245802
9	10, 12, 16, 18, 20, 24, 27, 28, 30, 32, 36, 40, 45, 48, 50, 54	
10	12, 18, 20, 21, 24, 27, 30, 36, 40, 42, 45, 48, 50, 54, 60, 63	A005349
11	12, 15, 20, 22, 24, 25, 30, 33, 35, 36, 40, 44, 45, 48, 50, 55	
12	22, 24, 33, 36, 44, 48, 55, 60, 66, 72, 77, 84, 88, 96, 99, 108	

Only 1, 2, 4, 6 are Niven numbers in every integer base $q > 1$. The number 12 is a Niven number in all bases except octal.

Density of Niven numbers

Let $N_q(x)$ denote the number of Niven numbers $\leq x$ in base q. Kennedy *et al.* proved that $\lim_{x\to\infty} N_{10}(x)/x = 0$, and that (1984) given any $t > 0$ we have $N_{10}(x) \geq \log^t x$.

Vardi proved that, given any $\varepsilon > 0$ we have $N_{10}(x) \ll x/(\log x)^{1/2-\varepsilon}$ and that $N_{10}(x) > \alpha x/(\log x)^{11/2}$ for some $\alpha > 0$ and infinitely many x.

De Koninck and Doyon proved that, given any fixed $\varepsilon > 0$ we have

$$x^{1-\varepsilon} \ll N_{10}(x) \ll \frac{x \log \log x}{\log x}$$

and conjectured that even $N_{10}(x) \sim \frac{cx}{\log x}$ with $c = \frac{14}{27} \log 10 \doteq 1.1939$. The conjecture has been verified by De Koninck, Doyon and Kátai in the form

$$N_q(x) = (\eta_q + o(1)) \frac{x}{\log x} \quad \text{with } \eta_q = \frac{2 \log q}{(q-1)^2} \sum_{j=1}^{q-1} (j, q-1).$$

Kennedy, R.E.: *Digital sums, Niven numbers, and natural density*, Crux Mathematicorum **8** (1982), 5, 129-133
Kennedy, R.E., Cooper, C.N.: *On the natural density of the Niven numbers*, The College Mathematics Journal **15** (1984), 4, 309-312
Cooper, C.N., Kennedy, R.E.: *Chebyshev's inequality and natural density*, Am. Math. Mon. **96** (1989), 2, 118-124
De Koninck, J.-M., Doyon, N.: *On the number of Niven numbers up to x*, Fibonacci Q. **41** (2003), 5, 431-440
De Koninck, J.-M., and Doyon, N., Kátai, I.: *On the counting function for the Niven numbers*, Acta Arith. **106** (2003), 3, 265-275
Vardi, I.: *Computational recreations in Mathematica*, Addison Wesley Publ. Comp., Redwood City, CA 1991 (pp. 28-30).

Another peculiar number connected with the sum-of-digits functions

The Canadian-American mathematician Albert "Tommy" Wilansky (1921-2017) of Lehigh University noticed that his brother-in-law Harold Smith, who is not a mathematician, observed that his phone number 493-7775 has the following remarkable property: *sum of its digits is equal to the sum of the digits of primes in its prime factorization in the same base*,

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Smith numbers in base 10: 1, 6, 49, 376, 3 294, 29 928, 278 411, 2 632 758, 25 154 060, 241 882 509, ... (sequence A104170 in the OEIS).

Smith numbers can be constructed from factored repunits (cf. Hoffman)

McDaniel: There are infinitely many k-Smith numbers n, that is numbers satifying $s_{10}(n) = k \sum_{p^\alpha || n} \alpha s_{10}(p)$.

Gardner, M.: *Penrose tiles to trapdoor ciphers ... and the return of Dr. Matrix*, Rev. ed., The Mathematical Association of America, Washington, DC 1997 (pp. 299-301).
Hoffman, P.: *The man who loved only numbers: the story of Paul Erdős and the search for mathematical truth*, Hyperion, New York 1998 (pp. 205-206).
McDaniel, W.L.: *The existence of infinitely many k-Smith numbers*, Fibonacci Q. **25** (1987), 1, 76-80
Wilansky, A.: *Smith numbers*, The Two-Year College Mathematics Journal **13** (1982), 21

Segments of Smith numbers

Two consecutive Smith numbers (for example, 728 and 729, or 2 964 and 2 965) are called **Smith brothers**. It is not known how many Smith brothers there are.

The starting elements of the smallest Smith *n*-tuple (meaning n consecutive Smith numbers) in base 10 for $n = 1, 2, \ldots$ are: 4, 728, 73 615, 4 463 535, 15 966 114, 2 050 918 644, 164 736 913 905, ... (sequence A059754 in the OEIS).

Other types of Smith Numbers

Fibonacci numbers, which are also Smith numbers

$$\begin{aligned} F_{31} &= 1\,346\,269 = 557 \cdot 2417 \\ F_{77} &= 5\,527\,939\,700\,884\,757 = 13 \cdot 89 \cdot 988681 \cdot 4832521 \\ F_{231} &= 844\,617\,150\,046\,923\,109\,759\,866\,426\,342\,507\,997\,914\,076\,076\,194 \\ &= 2 \cdot 13 \cdot 89 \cdot 421 \cdot 19\,801 \cdot 988\,681 \cdot 4\,832\,521 \cdot 9164\,259\,601\,748\,159\,235\,188\,401 \end{aligned}$$

Smith numbers, which are perfect squares, can be termed as **Smith Square Numbers** (A098839 OEIS).

Gupta, S.S.: *Smith numbers*, http://www.shyamsundergupta.com/smith.htm; Online; accessed June 6, 2023.

The lowest 3×3 Smith magic square ($C = 822$)

94	382	346
526	274	22
202	166	454

Gardner, M.: *Penrose tiles to trapdoor ciphers . . . and the return of Dr. Matrix*, Rev. ed., The Mathematical Association of America, Washington, DC 1997 (pp. 299-301).

Lucky tickets

Problem: Every bus ticket has an eight digits identification number. A ticket is called a **lucky** one if the sum of the first three its digits equals with sum of its right triple of digits. For instance, tickets with numbers 123006 or 777993, etc. are lucky ones. The question is: how many lucky tickets does exist?

a_n number of triples with sum n, $\quad a_n^2$ number of happy tickets with "sum" n

$A_1(s) = 1 + s + \cdots + s^9, \quad A_3(s) = (A_1(s))^3$

solution gives the absolute term of $P(s) = A_3(s)A_3(s^{-1})$

Cauchy theorem implies # of happy tickets$= \frac{1}{\pi}\int_{-\pi/2}^{\pi/2}\left(\frac{\sin 10\phi}{\sin\phi}\right)^6 \mathrm{d}\phi$

Ландо (Lando), С.К. (S.K.): Лекции о производящих функциях. *(Lectures on generating functions)*, Издание третье, исправленное (Third corrected edition), Издательство Московского центра непрерывного математического образования, Москва (Moscow) 2007

The generating functions

For $q \geq 2$ we have

$$\sum_{n\geq 0} s_q(n) z^n = \frac{1}{1-z} \sum_{m\geq 0} \frac{z^{q^m} + 2z^{2q^m} + \cdots + (q-1)z^{(q-1)q^m}}{1 + z^{q^m} + z^{2q^m} + \cdots + z^{(q-1)q^m}}$$

$$= \frac{1}{1-z} \sum_{m\geq 0} \frac{z^{q^m} - qz^{q^{m+1}} + (q-1)z^{(q+1)q^m}}{(1-z^{q^m})(1-z^{q^{m-1}})}$$

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The generating function for the sum-of-digits function of n written in the factorial (Cantor integer) base is

$$\frac{1}{1-z} \sum_{m\geq 0} \frac{z^{m!} + 2z^{m!} + \cdots + mz^{m\cdot m!}}{1 + z^{m!} + z^{2m!} + \cdots + z^{m\cdot m!}}$$

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$$= \frac{1}{1-z}\sum_{m\geq 0}\frac{z^{q^m}-qz^{q^{m+1}}+(q-1)z^{(q+1)q^m}}{(1-z^{q^m})(1-z^{q^{m-1}})}$$

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The generating function for the sum-of-digits function of n written in the generalized factorial multi-radix base $k_0 = 1 \times k_1 \times k_2 \times \ldots$ with $\bar{k}_j = k_0 k_1 \ldots k_{j-1}$ is

$$\frac{1}{1-z}\sum_{m\geq 0}\frac{z^{\bar{k}_m}+2z^{\bar{k}_m}+\cdots+(k_m-1)z^{(k_m-1)\bar{k}_m}}{z^{\bar{k}_m}+z^{2\bar{k}_m}+\cdots+z^{(k_m-1)\bar{k}_m}}$$

Adams-Watters, F.T., Ruskey, F.: *Generating functions for the digital sum and other digit counting sequences*, J. Integer Seq. **12** (2009), N0. 5, Article ID 09.5.6, 9 p.

Digit sums and q-series

If $B \geq 2$ (change of base notation), $z \in \mathbb{C}$ and $|q| < 1$ then

$$\sum_{n=0}^{\infty} q^n z^{s_B(n)} = \prod_{i=0}^{\infty} \frac{1 - z^B q^{B^{i+1}}}{1 - zq^{B^i}}$$

$$= \frac{1}{1-zq} + \frac{z - z^B}{1 - z^B q} \sum_{n=1}^{\infty} q^{B^n} \frac{\prod_{j=0}^{n-1}(1 - z^B q^{B_j})}{\prod_{j=0}^{\infty}(1 - zq^{B^j})}$$

q-series generating function for $s_B(n)$ with $B \geq 2$ and $|q| < 1$

$$\sum_{n=1}^{\infty} s_B(n) q^n = \frac{q}{(1-q)^2} - \frac{B-1}{1-q} \sum_{i=1}^{\infty} \frac{q^{B^i}}{1 - q^{B^i}}$$

Schneider, M., Schneider, R.: *Digit sums and generating functions*, Ramanujan J. **52** (2020), 2, 291-302

Lambert series generating function for $s_B(n)$ is the q-series generating function for $S_B(n) = \sum_{d|n} s_B(d)$

$$\sum_{n=1}^{\infty} \frac{s_B(n)q^n}{1-q^n} = \sum_{n=1}^{\infty} S_B(n)q^n$$

A Dirichlet convolution relation between Dirichlet series generating functions for $s_B(n)$ and $S_B(n)$

$$\zeta(s) \sum_{n=1}^{\infty} \frac{s_B(n)}{n^s} = \sum_{n=1}^{\infty} \frac{S_B(n)}{n^s}$$

(for the convergence of the series note that $1 \leq s_B(n) \leq s_{B'}(n) < n$ for $B < B'$)

Schneider, M., Schneider, R.: *Digit sums and generating functions*, Ramanujan J. **52** (2020), 2, 291-302

Happy numbers

A **happy number** is a number defined by the following process:

- Starting with any positive integer, replace the number by the sum of the squares of its (decimal) digits.
- Repeat the process until the number equals 1 (where it will stay), or it loops endlessly in a cycle which does not include 1.
- Those numbers for which this process ends in 1 are **happy**.

It is known [Honsberger, Porges] that eventually all the terms in the sequence are 1 or eventually the sequence becomes periodic with the cycle $4 \to 16 \to 37 \to 58 \to 89 \to 145 \to 42 \to 20 \to 4$.

If $S_{e,q}\left(\sum_{j\geq 0} a_j q^j\right) = \sum_{j\geq 0} a_j^e$ and $S_{e,q}^k(n) = S_{e,q}(S_{e,q}^{k-1}(n)) = 1$ for some $k \geq 0$ we say n is e-**power** q-**happy number** (Treviño *et al.* for fractional-base systems)

Honsberger, R.: *Ingenuity in mathematics*, New Mathematical Library. 23, 6th printing, Mathematical Association of America, Washington, DC. 1998 (pp. 74, 83–84).
Porges, A.: *A set of eight numbers*, Am. Math. Mon. **52** (1945), 379-382
Sánchez Garza, M., Treviño, E.: *On a sequence related to the factoradic representation of an integer*, J. Integer Seq. **24** (2021), 8, article 21.8.5, 13
Treviño, E., and Zhylinski, M.: *On generalizing happy numbers to fractional-base number systems*, Involve **12** (2019), 7, 1143-1151
Grundman, H.G., Hall-Seelig, L.L.: *Happy numbers, happy functions, and their variations: a survey*, La Matematica **1** (2022), 2, 404-430

What is a 'serious' problem?

Hardy: "A 'serious' theorem is a theorem which contains 'significant' ideas, and I suppose that I ought to try to analyse a little more closely the qualities which make a mathematical idea significant.

...

There are just four number (after 1) which are the sums of the cubes of their digits, vz.

$$153 = 1^3 + 5^3 + 3^3 \qquad 370 = 3^3 + 7^3 + 0^3$$
$$371 = 3^3 + 7^3 + 1^3 \qquad 407 = 4^3 + 0^3 + 7^3.$$

These are odd facts, very suitable for puzzle columns and likely to amuse amateurs, but there is nothing in them which appeals much to a mathematician. The proofs are neither difficult nor interesting — merely a little tiresome. The theorems are not serious; and it is plain that one reason (though perhaps not the most important) is the extreme speciality of both the enunciations and the proofs, which are not capable of any significant generalization...."

Hardy, G.H: *A Mathematician's Apology*, Cambridge University Press, Cambridge 1967 (pp. 103-105).

What is a 'serious' problem?

Hasse & Prichett: "This intriguing problem ...

Let $T(a)$ be a function defined on the rational integers which maps each positive rational integer a to the sum of the squares of its digits $s_{2,q}$. Proved that successive applications of T, commencing with any positive integer a, will always culminate in one of two possible cycles of integers ...

To discover the distinctive number-theoretic features of such problem it is far better to pose the question for all possible bases $q \geq 2$ and not to restrict consideration only to the special case $g = 10$. ..."

Hasse & Prichett studied the fixed points of $s_{2,q}$ for any fixed $q \geq 2$ and developed an algorithm for finding all the fixed points of $s_{2,q}$ based on the factorisation $q^2 + 1$ over the ring $\mathbb{Q}[\mathrm{i}]$. Their conjectural list $\{6, 10, 16, 20, 26, 40\}$ of q when $s_{2,q}$ has exactly two cycles is not complete, and the finiteness of the set is open.

Hasse, H., Prichett, G.: *A conjecture on digital cycles*, J. Reine Angew. Math. **298** (1978), 8-15

Let P be a positive integer-valued function on the positive integers $\mathbb{N}$ and let

$$F\left(\sum_{j\geq 0} a_i q^i\right) = \sum_{j\geq 0} P(a_j),$$

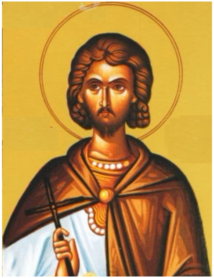
where the a_j are the digits of n expressed in base $q \geq 2$. For sufficiently large n we have $F(n) < n$. STEWART gives an efficient algorithm for finding the smallest C such that $F(n) < n$ for $n > C$. He studies growth properties of $F(n)$ with special emphasis on the case $P(n) = n^t$.

Zentralblatt 0098.26202: In the case $P(n) = n^t$ many ingenious methods are developed to find C.

If $P(a)$ is always a non-negative integer he investigates the orbit- and cycle-numbers resulting from the iteration of $F(n)$ and the finiteness of these numbers is assured.

STEWART, B.M.: *Sums of functions of digits*, Can. J. Math. **12** (1960), 374-389

Already Ancient Greeks



Saint Hippolytus of Rome (Mount Athos)

A form of sums of digits known to ancient Greek mathematicians was described by the Roman bishop Hippolytus (170–235) in *The Refutation of all Heresies*, and more briefly by the Syrian Neoplatonist philosopher Iamblichus (c.245–c.325) in his commentary on the *Introduction to Arithmetic* of Nicomachus of Gerasa.

Iamblichus Of Chalcis

Refutation catalogues both pagan beliefs and 33 gnostic Christian systems deemed heretical by Hippolytus.

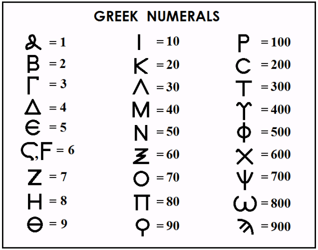
Hippolytus of Rome: *The Refutation of all Heresies*, in: *Ante-Nicene Fathers, Vol. V* (Roberts. A., Donaldson, J. Eds.), Scribner's Sons, New York 1919 (Book IV, Ch. 14, p. 30).

Heath, Th.: *A History of Greek Mathematics*, Vol. I: From Thales to Euclid, Oxford University Press, Oxford 1921 (p. 113-117).

Greek sums of digits via roots of numbers

There were two main systems of numerical notation in use in classical times in ancient Greece. Both were a decimal system.

- Attic numerals composed another system that came into use perhaps in the 7th century BCE. They were **acrophonic**, derived (after the initial one) from the first letters of the names of the numbers represented.
- The second main system, used for all kinds of numerals, is that with which we are familiar, namely the **alphabetic** system.



10 the unit of the second course: the **root** od **base** of 20 are two monads (pythmen)
100 the unit of the third course: the **root** of 600 are six monads
1000 the unit of the fouth course: the **root** of 700 are 7 monads

Sum of digits = sum of roots (monads)

Pythagorean calculus

Pythgoreans considered 10 as a unit of the second course, 100 as a unit of the third course, 1000 as a unit of fourth course, etc.

A Iamblichos proposition:

- Take the sum of three consecutive integers the greatest of which is divisible by 3, e.g. 10, 11, 12
- This sum consists of certain number of units, certain number of tens, certain number of hundreds, etc.In our case 3 units and 3 tens, i.e. 33
- Apply the procedure to the result, and so on.
- Iamblichos says: the final result will be number 6.

In *Refutation* we can find a description of a method of foretelling future by means of a calculation with numbers based on the notion of the monads (pythmen). It actually reduces to what we know as **gematria**, a practice of assigning a numerical value to a name, word or phrase according to an alphanumerical cipher used for notation of numbers. In decimal systems it actually reduces to the rule known as ‘casting out nines’.

Casting out nines – the Hindu Check

Important identity: $s_q(n) \equiv n \pmod{q-1}$, $q \geq 2$

When the decimal system was first employed (5th to 9th century CE), mathematicians recognized the fascinating properties of 9 and developed a time-honored rule of *casting out nines*. **Casting out nines** is an elementary check of a multiplication which makes use of the congruence $10^n \equiv 1 \pmod 9$.

Casting out nines was transmitted to Europe by the Arabs, but was probably developed in the ancient India. The earliest known surviving work which describes how casting out nines can be used to check the results of arithmetical computations is the *Mahâsiddhânta*, written around 950 by the Indian mathematician and astronomer Aryabhata II. (c. 920 - c. 1000).

Leonardo of Pisa (Fibonacci) introduced this rule to Medieval Europe through his book Liber abaci (1202) as a check for arithmetic operations.

de Moivre and the sum-of-digits

De Moivre's problem: In an urn there are g numbers $0, 1, 2, \ldots, g-1$. You carry out n draws one after the other, noting the number drawn each time, putting it back into the urn and mixing the contents of the urn. What is the probability that the sum of the digits drawn equals m?

Rohrbach: If $A^{(q)}_{n,m} = \mathrm{card}\{n : 0 \leq n \leq q^n, s_q(n) = m\}$ then an induction proof yields $A^{(q)}_{n,m} = \sum_{\nu=0}^{\infty} (-1)^\nu \binom{n}{\nu} \binom{m-q\nu+n-1}{n-1}$. The sum terminates when $m - q\nu < 0$.

Czuber, E.: *Wahrscheinlichkeitsrechnung und ihre Anwendung auf Fehlerausgleichung, Statistik und Lebensversicherung*, B.G. Teubner, Leipzig, Berlin 1914

Rohrbach, H.: *Die Anzahl der Zahlen mit vorgegebener Quersumme*, Math. Nachr. **1** (1948), 357-364

DE MOIVRE AND THE SUM-OF-DIGITS

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CHEO & YIEN: $\sum_{\substack{s_q(n)=m \\ n \le x}} 1 \sim \frac{1}{m!} \cdot \left(\frac{\log x}{\log q}\right)^m$

A saddle-point estimate for $\mathrm{card}\{n \le N : s_q(n) = m\}$ is given by MAUDUIT & SÁRKÖZY

CHEO, P-H., YIEN, S-CH.: *A problem on the k-adic representation of positive integers*, (Chinese), Acta Math. Sin. **5** (1955), 4, 433-438

MAUDUIT, CH., SÁRKÖZY, A.: *On the arithmetic structure of the integers whose sum of digits is fixed*, Acta Arith. **81** (1997), 2, 145–73

Some fascinating identities$_1$

Show that the sequence of increasingly complex fractions approaches a limit, and find that limit

$$\frac{1}{2},\ \frac{\frac{1}{2}}{\frac{3}{4}},\ \frac{\frac{\frac{1}{2}}{\frac{3}{4}}}{\frac{\frac{5}{6}}{\frac{7}{8}}},\ \frac{\frac{\frac{\frac{1}{2}}{\frac{3}{4}}}{\frac{\frac{5}{6}}{\frac{7}{8}}}}{\frac{\frac{\frac{9}{10}}{\frac{11}{12}}}{\frac{\frac{13}{14}}{\frac{15}{16}}}},\ \dots$$

Woods, D.R.: *Elementary problem E2692*, Amer. Mat. Monthly **48** (1978), 1, 48
Woods, D.R., David Robbins, D., Gustaf Gripenberg, G.: *Solution of E2692*, Amer. Mat. Monthly **86** (1979), 5, 394-395

Some fascinating identities[1]

Show that the sequence of increasingly complex fractions approaches a limit, and find that limit

$$\frac{x}{x+1},\ \frac{\frac{x}{x+1}}{\frac{x+2}{x+3}},\ \frac{\frac{\frac{x}{x+1}}{\frac{x+2}{x+3}}}{\frac{\frac{x+4}{x+5}}{\frac{x+6}{x+7}}},\ \frac{\frac{\frac{\frac{x}{x+1}}{\frac{x+2}{x+3}}}{\frac{\frac{x+4}{x+5}}{\frac{x+6}{x+7}}}}{\frac{\frac{\frac{x+8}{x+9}}{\frac{x+10}{x+11}}}{\frac{\frac{x+12}{x+13}}{\frac{x+14}{x+15}}}},\ \ldots$$

Woods, D.R.: *Elementary problem E2692*, Amer. Mat. Monthly **48** (1978), 1, 48
Woods, D.R., David Robbins, D., Gustaf Gripenberg, G.: *Solution of E2692*, Amer. Mat. Monthly **86** (1979), 5, 394-395

Some fascinating identities$_1$

If $k \geq 2$ is an integer and $1 \leq j \leq k-1$ then

$$\prod_{i=0}^{\infty} \frac{1+c_i}{1+d_i} = k^{-1/k}$$

The c_i and d_i are such that $ki \leq c_i, d_i < k(i+1)$, $s_k(c_i) \equiv j-1 \pmod{k}$ and $s_k(d_i) \equiv j \pmod{k}$). For example, if $k=2$ and $j=1$ we obtain the following infinite product:

$$\frac{1}{2} \cdot \frac{4}{3} \cdot \frac{6}{5} \cdot \frac{7}{8} \cdots = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

Shallit, J.O.: *On infinite products associated with sums of digits*, J. Number Theory **21** (1985), 128-134

Some fascinating identities$_1$

If k is an even positive integer and c_i and d_i are such that $2i \le c_i, d_i < 2(i+1)$, $s_k(c_i) \equiv 0 \pmod 2$ and $s_k(d_i) \equiv 1 \pmod 2$), then

$$\prod_{i=0}^{\infty} \frac{1+c_i}{1+d_i} = \frac{\sqrt{k}}{k}$$

For example, if $k = 6$ we obtain the following infinite product:

$$\frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdot \frac{8}{7} \cdots = \frac{\sqrt{6}}{6}$$

Shallit, J.O.: *On infinite products associated with sums of digits*, J. Number Theory **21** (1985), 128-134

How independent are s_q's?

Remember the Hindu check $s_q(n) \equiv n \pmod{q-1}$, $q \geq 2$.

If we understand the digits as random variables $x_I, x_2, \dots, x_n$ which take independently the values $0, 1, \dots, q-1$, then it follows from the main principle of probability theory that the distribution of the new random variables

$$p = x_1 + x_2 + \cdots + x_n,$$

as n increases, tends to a normal (Gaußian) distribution with the mean $(g-1)n/2$ and variance $(g^2-1)n/12$ and the probability function

$$W_{n,m}^{(q)} = \frac{\exp\left(-\frac{6\left[m-\frac{1}{2}(q-1)n\right]^2}{(q^2-1)n}\right)}{\sqrt{\frac{1}{6}(q^2-1)n}}$$

The proof uses the generating function technique introduced by De Moivre employing the generating function $(1 + x + \cdots + x^{q-1})^n = \left(\frac{1-x^q}{1-x}\right)^n$.

Rohrbach, H.: *Die Anzahl der Zahlen mit vorgegebener Quersumme*, Math. Nachr. **1** (1948), 357-364

How independent are s_q's?

S. Ulam has asked whether the number of $n < x$ for which $s_{10}(n) \equiv n \equiv 0 \pmod{13}$ is asymptotically $x/13^2$?

$N_{a,c,p}(x) = \mathrm{card}\{n < x : n \equiv a \pmod{p} \text{ and } s_q(n) \equiv c \pmod{p}\}$

Let $a, c \in \mathbb{N}_0$ and p be a prime such that $p \nmid (q-1)$. Then

$$\lim_{x \to \infty} \frac{N_{a,c,p}(x)}{x} = \frac{1}{p^2}.$$

Fine remarks (without proof) that for distinct primes p, q, the residues of $n \pmod{p}$ and $s_q(n) \pmod{q}$ are asymptotically independent.

Fine, N.J.: *The distribution of the sum of digits* $\pmod{p}$, Bull. Amer. Math. Soc. **71** (1965), 651-651

How independent are s_q's?

$V_k(x) = \mathrm{card}\{n : 0 \leq n < x, s_q(n) = k\}$

$V_k(x; m, h) = \mathrm{card}\{n : 0 \leq n < x, s_q(n) = k, n \equiv h \pmod m\}$

There exist positive numbers ℓ_1, c_0, c_1, c_2, all depending on q alone, such that if $x > 1$ is a real number, m is a positive integer with $(m, q) = 1$, k, h, ℓ are integers such that $\ell > \ell_1$, and $m < \exp(c_0\sqrt{\ell})$, then, writing $d = (m, q-1)$, we have

$$\left| V_k(x; m, h) - \frac{d}{m} V_k(x) \right| < \frac{c_1}{m} V_k(x) \exp(-c_2\ell/\log m)$$

for $k \equiv h \pmod d$ and

$$V_k(x; m, h) = 0$$

if $k \not\equiv h \pmod d$.

Mauduit, Ch., Pomerance, C., Sárközy, A.: *On the distribution in residue classes of integers with a fixed sum of digits*, Ramanujan J. **9** (2005), 1-2, 45-62

Some fascinating identities$_2$

Shallit: $\sum_{n=1}^{\infty} \frac{s_q(n)}{n(n+1)} = \frac{q}{q-1} \log q$,

$$\sum_{n\geq 1} \frac{s_2(n)}{2n(2n+1)(2n+2)} = -\frac{1}{2}\log\pi + \frac{\gamma}{2} + \frac{1}{2}\log 2$$

Allouche & Shallit $(q=2)$: If $\Re(s) > 0$ then

$$\sum_{n=1}^{\infty} s_q(n)\left(\frac{1}{n^s} - \frac{1}{(n+1)^s}\right) = \frac{q^s - q}{q^s - 1}\zeta(s)$$

If $a_{w,q}(n)$ denote the number of (possibly overlapping) occurrences of the word w in the q-ary expansion of n then

$\sum_{n=1}^{\infty} a_{w,q}(n)\left(\frac{1}{n^s} - \frac{1}{(n+1)^s}\right)$ is expressible in terms of Hurwitz zeta-functions

A variety of generalizations in:

Allouche, J.-P., Shallit, J., Sondow, J.: *Summation of series defined by counting blocks of digits*, J. Number Theory **123** (2007), 1, 133-143
Vignat, Ch., Wakhare, T.: *Finite generating functions for the sum-of-digits sequence*, Ramanujan J. **50** (2019), 3, 639-684

Erdős – Kac behaviour

If $\omega(n)$ denotes the number of prime divisors of n then (Erdős & Kac)

$$\lim_{n\to\infty} \frac{1}{n} \operatorname{card}\{n \le x : \omega(n) < \log\log n + \lambda\sqrt{2\log\log n}\} = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \exp(-u^2)\mathrm{d}u$$

with λ an arbitrary real number.

Let $m \in \mathbb{N}$, $(m, q-1) = 1$ and $U_r(N) = \{n \le N : s_q(n) \equiv r \pmod{m}\}$. Then

$$\lim_{N\to\infty} \frac{\operatorname{card}\left\{n \in U_r(N) : \frac{\omega(n)-\log\log N}{\sqrt{\log\log N}}\right\}}{\operatorname{card}(U_r(N))} \to \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{X} \exp\left(-u^2/2\right) \mathrm{d}u$$

uniformly in X.

Erdős, P., Kac, M.: *The Gaussian Law of Errors in the Theory of Additive Number Theoretic Functions*, Amer. J. Math. **62** (1940), 1, 738-742
Mauduit, Ch., Sárközy, A.: *On the arithmetic structure of the integers whose sum of digits is fixed*, Acta Arith. **81** (1997), 2, 145-173

BUSH: $\frac{1}{x}\sum_{n\leq x)} s_q(n) \sim \frac{q-1}{2\log q}\log x$ as $x\to\infty$

BUSH thus proved a statement given in BOWDEN's book (p. 68) for which the author had no general proof, namely that the average sum of the digits of integers is least when they are written in the binary scale.[2]

BELLMAN & SHAPIRO: $\frac{1}{x}\sum_{n\leq x} s_2(n) = \frac{\log x}{2\log 2} + O(\log\log x)$ and claimed that one of them improved the error term to $O(1)$, what is the best possible result. This was proved definitely by MIRSKY.

GADD & WONG noticed a mistake in the proof of BELLMAN & SHAPIRO thought their result is correct, and proved $\sum_{n\leq x} s_q(n) = \frac{\log x}{2\log q} + O(\log\log x)$ for all integer bases $q\geq 2$.

FANG: $\sum_{n\leq x} s_q(n) = \frac{q-1}{2\log q}x\log x + \theta(x)x$, where $-\frac{5q-4}{8}\leq\theta(x)\leq\frac{q+1}{2}$

BELLMAN, R., SHAPIRO, H.N.: *On a problem in additive number theory*, Ann. Math. (2) **49** (1948), 333-340
BOWDEN, J.: *Special topics in theoretical arithmetic*, Garden City, New York 1936
BUSH, L.E.: *An asymptotic formula for the average sum of the digits of integers*, Am. Math. Mon. **47** (1940), 154-156
FANG, Y.: *A theorem on the k-adic representation of positive integers*, Proc. Amer. Math. Soc. **130** (2002), 6, 1619-1622
GADD, C., WONG, K.L.: *A generalization to Bellman and Shapiro's method on the sum of digital sum functions*, PUMP J. Undergrad. Res. **5** (2022), 176-187
MIRSKY, L.: *A theorem on representations of integers in the scale of r*, Scripta Math. **15** (1949), 11-12

[2] BOWDEN on pages 17–81 devoted to scales of numeration wished to abolish the "tyranny of ten" (p. 81) and proposes, among other things, to have a 16-hour day (p. 77).

$N = \sum_{j=1}^{t} a_j q^{n_j}$ with $n_1 > n_2 > \ldots n_t \geq 0$ and $1 \leq a_j \leq q - 1$

$$\sum_{n \leq N} s_q(n) = \underbrace{\frac{n_1(q-1)}{2} \sum_{j=1}^{t} a_j q^{n_j}}_{\frac{q-1}{2}\frac{N \log N}{\log q}} -$$

$$\underbrace{\frac{q-1}{2} \sum_{j=1}^{t} (n_1 - n_j) a_j q^{n_j} + \frac{1}{2} \sum_{j=1}^{t} a_j (a_j - 1) q^{n_j} + \sum_{j=1}^{t} a_j + \sum_{j=1}^{t} \left(\sum_{k=1}^{j-1} a_k \right) a_j q^{n_j}}_{O(x)}$$

CHEO, P-H., YIEN, S-CH.: *A problem on the k-adic representation of positive integers*, (Chinese), Acta Math. Sin. **5** (1955), 4, 433-438

In 1975 DELANGE proved a prototype result without the error term

$$\sum_{n\le N} s_q(n) = \frac{q-1}{2} N \log_q N + NF(\log_q N),$$

where $\log_q$ stands for the logarithm to base q and F is a 1-periodic, continuous and nowhere differentiable function explicitly given in three steps:

First, he defined the function g on $\mathbb{R}$ by the formula

$$g(x) = \int_0^x \left(\lfloor qt \rfloor - q\lfloor t \rfloor - \frac{q-1}{2} \right) \mathrm{d}t$$

then the function

$$h(x) = \sum_{r=0}^{\infty} \frac{g(q^r x)}{q^r}$$

and finally

$$F(x) = \frac{q-1}{2}(1 + \lfloor x \rfloor - x) + q^{1+\lfloor x \rfloor - x} h(q^{1+\lfloor x \rfloor - x}).$$

DELANGE, H.: *Sur la fonction sommatoire de la fonction 'somme des chiffres'*, Enseign. Math. (2) **21** (1975), 31-47
MADRITSCH, M.G.: *The summatory function of q-additive functions on pseudo-polynomial sequences*, J. Théor. Nombres Bordx. **24** (2012), 1, 153-171

DELANGE actually generalized an explicit result for the remainder term proved by TROLLOPE for binary expansion:

$$2^{m-1}\big(2f(x) + (1+x)\log_2(1+x) - 2x\big)$$

where the integer N is written in the form $N = 2^m(1+x)$ with $0 \le x < 1$ and $f(x) = \sum_{i=0}^{\infty} \frac{g(2^i x)}{2^i}$ with

$$g(x) = \begin{cases} \frac{1}{2}x, & \text{if } 0 \le x \le \frac{1}{2} \\ \frac{1}{2}(1-x), & \text{if } \frac{1}{2} < x \le 1. \end{cases}$$

TROLLOPE also proved the estimate for the error term

$$2^{m-1}\left(\frac{5}{3}\log_2\frac{5}{3} - \frac{2}{3}\right)$$

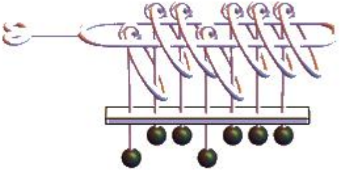
where constant cannot be reduced.

TROLLOPE, J.R.: *An explicit expression for binary digital sums*, Math. Mag. **41** (1968), 21-25

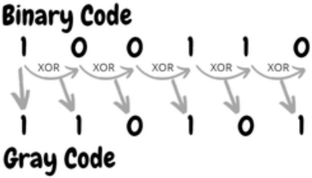
Flajolet and Ramshaw showed that, Delange's proof method for computing the sum of all of the digits used when the first n nonnegative integers are expressed in base q, can be adapted to some 'unusual' number systems as Gray code or balanced ternary and its generalization or even can also be adapted to count the occurrences of each digit separately.

Flajolet, P. and Ramshaw, L.: *A note on Gray code and odd-even merge*, SIAM J. Comput. **9** (1980), 142-158

Gray code



0110111



Wikipedia

In Gray code each number from $\{0, 1, \dots, 2^N - 1\}$ is represented as the sequence of integers as a binary string of length N in an order in which the adjacent integers have Gray code representations differing in only one bit position (OEIS A014550).

In essence, a Gray code takes a binary sequence and shuffles it to a new form sequence with the mentioned adjacency property.

Binary	000	001	010	011	100	101	110	111
Gray	000	001	011	010	110	111	101	100

Heeffer, A., Hinz, A.M.: *"A difficult case": Pacioli and Cardano on the Chinese Rings*, Recreational Mathematics Magazine **4** (2017), 8, 5-23
Stibitz, G.R.: *Binary counter*, Bell Telephone Laboratories, Incorporated. U.S. Patent 2,307,868. Serial No. 420537, 1943
Gray, F.: *Pulse Code Communication*, U. S. Patent 2 632 058 March 17, 1953

Gray code and sum-of-digits function

Let $\gamma(n)$ denote the number of 1-bits in the standard Gray code representation of n. There exists a continuous, nowhere differentiable function $G : \mathbb{R} \to \mathbb{R}$, periodic with period 1, such that

$$\sum_{n \le N} \gamma(n) = \frac{N \log_2 N}{2} + NG(\log_2 n)$$

The Fourier series representation $G(x) = \sum_k g_k \exp(2k\pi \mathrm{i} x)$ converges absolutely and its coefficients g_k are given explicitly:

$$g_0 = 2 \log_2 \Gamma\left(\frac{1}{4}\right) - \log_2 \pi - \frac{1}{2 \log_2 2} - \frac{5}{4}$$

$$g_k = \frac{2\zeta(\chi_k, \frac{1}{4})}{(\log_2 2)\chi_k(1 + \chi_k)}, \qquad \chi_k = \frac{2k\pi\mathrm{i}}{\log_2 2}, \quad \zeta(z, \alpha) = \sum_{j \ge 0} (j + \alpha)^{-z}.$$

Gardner, M.: *Mathematical games*, Scientific American **227** (1972), 2, , 106-109
Flajolet, P. and Ramshaw, L.: *A note on Gray code and odd-even merge*, SIAM J. Comput. **9** (1980), 142-158
Larcher, G., Tichy, R.F.: *A note on Gray code and odd-even merge*, Discrete Appl. Math. **18** (1987), 309-313

Balanced (q, r)-ary number systems

A balanced ternary number system is a numeral system that comprises digits $-1, 0$, and 1.

$464_{10} = 1 \cdot 3^5 + 2 \cdot 3^4 + 2 \cdot 3^3 + 1 \cdot 3 + 2 \cdot 1 = 122012_3.$

$464_{10} = 1 \cdot 3^6 - 1 \cdot 3^5 - 1 \cdot 3^3 + 1 \cdot 3^2 - 1 \cdot 3 - 1 \cdot 1 = R_1 L_1 0 L_1 R_1 L_1 L_1$

Most famous application: The weight problem of Bachet de Méziriac when the weight can be placed in either pan of the balance.

Balanced (q, r)-ary number systems

Balanced q-ary number system:

- If q is odd, say $q = 2r + 1$, $r = 1, 2, 3, \ldots,$ one can express any natural number N in base (q, r) in terms of symbols $-r, -r + 1, \ldots, 0, 1, \ldots, r - 1, r$ in symbols $0, L_i, R_i,\ i = 1, 2, \ldots, , r$
- If q is even, say $q = 2r,\ r = 1, 2, 3, \ldots,$ one can express any natural number N in base (q, r) in terms of symbols $-r + 1, \ldots, 0, 1, \ldots, r - 1$ in symbols $0, L_i, R_i,\ i = 1, 2, \ldots, , r - 1$ and for $-r$ with L_r (or for $r = m$ with R_r).

Balanced (q, r)-ary number systems

More generally, digits need not be 'centered' around 0. We can take q consecutive integers including 0 as digits. Denote by (q, r) **number system**, where q denotes the base and r, $0 \leq r \leq q - 1$, denotes the number of negative digits $-r, 1 - r, \ldots, \ldots, q - 1 - r$.

- balanced ternary system is the $(3, 1)$ system
- q-ary number system is $(q, 0)$ system

The number of d-digits in the (q, r) positional number system

Let q and r be integers satisfying $q \geq 2$ and $0 \leq r \leq q - 2$. Let the (q, r) number system be the positional number system with base q and digits $-r, 1 - r, q - 1 - r$, and let d be a non-zero digit in this system. Let $\rho(n)$ denote the number of times that the digit d is used when n is expressed in the (q, r) number system, and let $F(\rho, n)$ denote the appropriately truncated summation of ρ, in particular,

$$F(d, n) = \left(1 - \frac{r}{q-1}\right)\rho(0) + \rho(1) + \rho(2) + \cdots + \rho(n-1) + \frac{r}{q-1}\rho(n).$$

Then, there exists a continuous, nowhere differentiable function $P : \mathbb{R} \to \mathbb{R}$, periodic with period 1, such that

$$F(d, n) = \frac{n \log_q n}{q} + nP(\log_q n) \quad \text{for } n \geq 1$$

The Fourier series $P(x) = \sum_k p_k exp(2k\pi \mathrm{i}x)$ converges absolutely, and the coefficients p_k are given expolicitely.

Flajolet, P. and Ramshaw, L.: *A note on Gray code and odd-even merge*, SIAM J. Comput. **9** (1980), 142-158

The Zeckendorf sum-of-digits function

The **Zeckendorf decomposition** of a natural number n is the unique expression of n as a sum of Fibonacci numbers with non-consecutive indices and with each index greater than 1:

$$309\,018 = 196\,418 + 75\,025 + 28\,657 + 6\,765 + 1\,597 + 377 + 144 + 34 + 1 = F_{27}+F_{25}+F_{23}+F_{17}+F_{14}+F_{12}+F_{9}+F_{1} = (1010100100100101001000001)_{\mathrm{Zeck}}$$

Let n be a positive integer. Define $s_{\mathrm{Zeck}}(n)$ as the sum (number) 1's in the Zeckendorf decomposition of the natural number n. We have (Coquet & van den Bosch)

$$\sum_{n<x} s_{\mathrm{Zeck}}(n) = \frac{3-\beta}{5\log\beta} x\log x + xG\left(\frac{\log x}{\log\beta}\right) + O(\log x)$$

(G is a real valued continuous, nowhere differentiable function of period 1 and $\beta = (1+\sqrt{5})/2$)

Coquet, J., van den Bosch, P.: *A summation formula involving Fibonacci digits*, J. Number Theory **22** (1986), 139-146
Zeckendorf, E.: *Représentation des nombres naturels par une somme de nombres de Fibonacci ou de nombres de Lucas*, Bull. Soc. R. Sci. Liège **41** (1972), , 179-182

k-Zeckendorf representation

The k-Zeckendorf representation of a positive integer n is defined as the sum of the k-generalized Fibonacci numbers $n = \sum_{i \geq k} \varepsilon_i F_i^{(k)}$, where $\varepsilon_i \in \{0, 1\}$ and for all $i \geq k$ we have $\varepsilon_i \varepsilon_{i+1} \cdots \varepsilon_{i+k-1} = 0$.

The k-generalized Fibonacci sequence for $k = 2, 3, 4, 5, 6, 7, 8$ can be found in OEIS as sequences A000045, A000073, A000078, A001591, A001592, A122189, and A079262, respectively.

Given this representation of a number n we say the k-Zeckendorf digital sum of n is $s_{k\text{-Zeck}}(n) = \sum_{i \geq k} \varepsilon_i$ and if $s_{k\text{-Zeck}}(n) \mid n$ then n is called a **k-Zeckendorf Niven number**.

For instance, every $F_i^{(k)}$ is a k-Zeckendorf Niven number or 8 and 12 are 3-Zeckendorf Niven numbers.

The asymptotic density of the k-Zeckendorf Niven numbers is zero.

Cooper, C.: *The k-Zeckendorf array*, in: *Proceedings of the 14th international conference on Fibonacci numbers and their applications, Morelia, Mexico, July 5–9, 2010*, (Luca, F., Stănică, P. Eds.), pp. 79-90, Sociedad Matemática Mexicana, México 2011,
Ray, A., Cooper, C.: *On the natural density of the k-Zeckendorf Niven numbers*, J. Inst. Math. Comput. Sci., Math. Ser. **19** (2006), 1, 83-98

Expansions and linear recurrences

Let $\mathcal{G} = (G_k)_{k\geq 0}$ be a linear recurring sequence

$$G_{k+d} = a_1 G_{k+d-1} + \cdots + a_d G_k$$

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$$G_{k+d} = a_1 G_{k+d-1} + \cdots + a_d G_k$$

with

- integral coefficients $a_1 \geq a_2 \geq \cdots \geq a_d > 0$, and
- integral initial values $1 = G_0, \ldots, G_{d-1}$ satisfying $a_1 > 1$ (for $d = 1$) and for $d \geq 2$ $G_k \geq a_1 G_{k-1} + \cdots + a_n G_0 + 1$ for $n = 1, \ldots, d-1$.

Expansions and linear recurrences

Let $\mathcal{G} = (G_k)_{k\geq 0}$ be a linear recurring sequence

$$G_{k+d} = a_1 G_{k+d-1} + \cdots + a_d G_k$$

Then every positive integer n can be represented in a unique way by $\sum_j \varepsilon_j(n) G_j$, where the G-ary digits $\varepsilon_j(n)$ are integers with $0 \leq \varepsilon_j < a_1$ satisfying some additional conditions (cf. Theorem 1 of Pethő & Tichy).

Then for the sum-of-digits function $s_{\mathcal{G}}(n) = \sum \varepsilon_j(n)$ with respect to the given linear recurrence $\mathcal{G}$ we have

$$\sum_{n<N} s_{\mathcal{G}}(n) = c_{\mathcal{G}} N \log N + N \cdot F\left(\frac{\log N}{\log \alpha}\right) + O(\log N),$$

where α is the dominating root of the characteristic polynomial of $\mathcal{G}$, $c_{\mathcal{G}}$ a suitable constant, and F is a bounded periodic function of period 1 (not necessary continuous, for more details Sec. 4 of Pethő & Tichy).

Pethő, A., Tichy, R.F.: *On digit expansions with respect to linear recurrences*, J. Number Theory **33** (1989), 2, 243-256

Non-integer base of numeration

A positional numeral system with a non-integer number $\beta > 1$ as the radix. If

$$x = \beta^n d_n + \cdots + \beta^2 d_2 + \beta d_1 + d_0 + \beta^{-1} d_{-1} + \beta^{-2} d_{-2} + \cdots + \beta^{-m} d_{-m}$$

then

$$x = (d_2 d_1 d_0 . d_{-1} d_{-2} \ldots d_{-m})_\beta$$

For instance, base $\sqrt{2}$ behaves in a very similar way to base 2. Every integer can be expressed in base $\sqrt{2}$ without the need of a decimal point (just put a zero digit in between every binary digit):

$$(5118)_{10} = (1001111111110)_2 = (1000001010101010101010100)_{\sqrt{2}}.$$

Bergman, G.: *A number system with an irrational base*, Math. Mag. **31** (1957), 98-110
Parry, W.: *On the β-expansions of real numbers*, Acta Math. Acad. Sci. Hung. **11** (1960), 3-4, 401-416
Rényi, A.: *Representations for real numbers and their ergodic properties*, Acta Math. Acad. Sci. Hung. **8** (1957), 3-4, 477-493

Golden mean representations

Base $\varphi = (1 + \sqrt{5}/2)$ was introduced by BERGMAN. A positive integer n written in base φ has the form $n = \sum_{j=0}^{\infty} \varepsilon_j \varphi^j$, with digits $\varepsilon_j \in \{0, 1\}$, and where $\varepsilon_j \varepsilon_{j+1} = 11$ is not allowed. Ignoring leading and trailing 0's, the base phi representation of a number n is unique.

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$$\begin{aligned}1\,000\,000 = {} & \varphi^{28} + \varphi^{26} + \varphi^{20} + \varphi^{16} + \varphi^{13} + \varphi^{8} + \varphi^{4} + \varphi^{0} + \\ & \varphi^{-4} + \varphi^{-9} + \varphi^{-11} + \varphi^{-14} + \varphi^{-16} + \varphi^{-20} + \varphi^{-26} + \varphi^{-28} = \\ & (10100000100010100101000010001.0001000100001001000100000101)_{\varphi}\end{aligned}$$

Golden mean representations

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If $s_\varphi(n) = \sum_j \varepsilon_j$, then $(s_\varphi(n))_{n\geq 0} = 0, 1, 2, 2, 3, 3, 3, 2, 3, 4, 4, 5, 4, 4, 4, 5, 4, 4, \ldots$

If L_n is the Lucas sequence: $L_0 = 2$, $L_1 = 1$, $L_i = L_{i-1} + L_{i-2}$ for $i \geq 2$ then (Cooper & Kennedy)

$$\sum_{k\leq L_n} s_\varphi(k) = \frac{3}{2} - \frac{3}{2}(-1)^n + \frac{1-\sqrt{5}}{2}\left(\frac{1+\sqrt{5}}{2}\right)^n + \frac{1+\sqrt{5}}{2}\left(\frac{1-\sqrt{5}}{2}\right)^n + \\ + \frac{5-\sqrt{5}}{2}\left(\frac{1+\sqrt{5}}{2}\right)^n (n+1) + + \frac{5+\sqrt{5}}{2}\left(\frac{1-\sqrt{5}}{2}\right)^n (n+1)$$

Bergman, G.: *A number system with an irrational base*, Math. Mag. **31** (1957), 98-110
Cooper, C.N., Kennedy, R.E.: *The first moment of the number of 1's function in the beta-expansion of the positive integers*, Journal of Institute of Mathematics & Computer Sciences **14** (2001), 69-77
Dekking, F.M.: *The sum of digits functions of the Zeckendorf and the base phi expansions*, Theor. Comput. Sci. **859** (2021), 70-79

Complex integer bases

The notion of congruence can be applied to any ring of algebraic integers $\mathbb{Z}[\beta]$ in an algebraic number field, modulo the element β in the ring. This ring $\mathbb{Z}[\beta]$ is isomorphic to the quotient ring $\mathbb{Z}[x]/(m(x))$, where $m(x)$ is the minimum polynomial of β.

Sylvester a.k.a. Lanavicensis: If β is a non-zero algebraic integer of norm $|\beta| = N$, then a complete residue system of elements of $\mathbb{Z}[\beta]$ modulo β contains $|N|$ elements and

$$\frac{\mathbb{Z}[\beta]}{\beta} \approx \mathbb{Z}_N.$$

Kátai & Szabó: Let β be a Gaussian integer of norm N and let $D = \{0, 1, 2, \ldots, N-1\}$. Then β is a valid base for the complex numbers using the digit set D if and only if $\beta = -a \pm i$ for some positive integer a.

Lanavicensis: *Note on complex integers*, Quart. J. Pure and Applied Math. **4** (1861), 94-96, 124-130
Kátai, I., Szabó, J.: *Canonical number-systems for complex integers*, Acta Sci. Math. **37** (1975), , 255-260

Complex integer bases

The only complex integer bases in $\mathbb{Z}[i]$, which give rise to a unique unite digital representation of the Gaussian integers using a 'connected' set of digits from the natural numbers (in our instance we have the digits $(0,\ldots,a^2)$), is based on Gaussian integers $a+\mathrm{i}$ with $a\in\mathbb{N}$. Let

$$z=\sum_{k=0}^{K}\varepsilon_k(-a+\mathrm{i})^k,\qquad \varepsilon_k\neq 0,\quad s_{-a+\mathrm{i}}(z)=\sum_{k=0}^{K}\varepsilon_k.$$

For instance, $-3+3i=(11010)_{-1+i}$

$$\sum_{|-2+\mathrm{i}|<N} s_{-a+\mathrm{i}}(z)=2\pi N\log_5 N+N\Phi(\log_5 N)+O(\sqrt{N}\cdot\log N)$$

(the proof could be extended to the general case)

Grabner, P.J., Kirschenhofer, P., Prodinger, H.: *The sum-of-digits function for complex bases*, J. Lond. Math. Soc., II. Ser. **57** (1998), 1, 20-40

Knuth, D.E.: *The art of computer programming. Vol. 2: Seminumerical algorithms*, Vol. 2, 3rd Ed., Addison-Wesley, Bonn 1998

SUM-OF-DIGITS FUNCTION ALONG THE REAL LINE

The summatory function $S(N)$ of the sum-of-digits function in base $q = -a + \mathrm{i}$, where $a = 1$ or $a \geq 2$ and even, of the first N positive integers satisfies

$$\frac{a^2}{2} \leq \liminf_{N\to\infty} \frac{S(N)}{N \log_{a^2+1} N} \leq \limsup_{N\to\infty} \frac{S(N)}{N \log_{a^2+1} N} \leq \frac{3a^2}{2}$$

GRABNER, P.J., KIRSCHENHOFER, P., PRODINGER, H.: *The sum-of-digits function for complex bases*, J. Lond. Math. Soc., II. Ser. **57** (1998), 1, 20-40

LEGENDRE'S FORMULAE & KUMMER'S THEOREM

A.-M. LEGENDRE: … En général, si on a $N = \theta^n$, le nombre de fracteurs θ compris dans le produit $1, 2, 3, \ldots, N$ sera

$$x = \frac{N-1}{\theta - 1}.$$

Et si on fait, comme on peut toujours la supposer,

$$N = A\theta^m + B\theta^n + C\theta^r + \text{etc.},$$

les coefficiens A, B, C,etc. étant plus petits que θ, il résulters

$$x = \frac{N - A - B - C - \text{etc.}}{\theta - 1}$$

LEGENDRE, A.-M.: *Essai sur la théorie des nombres*, 2de édition, Courcier, Paris 1808 (p. 10).

LEGENDRE'S FORMULAE & KUMMER'S THEOREM

Let $\nu_p(n)$ be the exponent of the largest power of prime p that divides n, then

$$\nu_p(n!) = \frac{n - s_p(n)}{p-1}.$$

Legendre's formulae & Kummer's theorem

Let $\nu_p(n)$ be the exponent of the largest power of prime p that divides n, then

$$\nu_p(n!) = \frac{n - s_p(n)}{p-1}.$$

Kummer:

$$\nu_p\left(\binom{n}{m}\right) = \frac{s_p(m) + s_p(n-m) - s_p(n)}{p-1}$$

Kummer's algorithm: the exact power of prime p that divides the binomial coefficient $\binom{n}{m}$ is given by the number of 'carries' when we add m and $n-m$ in base p.

Example: $3^3 \Big| \binom{189}{78}$ and $78 = (2220)_3$, $189 - 78 = 111 = (11010)_3$

Kummer, E.E.: *Über die Ergänzungssätze zu den allgemeinen Reciprocitätsgesetzen*, J. Reine Angew. Math. **44** (1852), 93-146 (p. 116).

If $n = (n_k n_{k-1} \ldots n_0)_q$ and $m = (m_k m_{k-1} \ldots m_0)_q$ with $n \geq m$, the **base q carries** when adding m and $n - m$ are defined by $\epsilon_{-1}^{n,m,q} = 0$ and for $i \geq 0$,

$$\epsilon_i^{n,m,q} = \begin{cases} 1, & \text{if } m_i > n_i \\ 1 & \text{if } m_i = n_i \text{ and } \epsilon_{i-1}^{n,m,q} = 1 \\ 0, & \text{otherwise.} \end{cases}$$

If $\kappa_q(m,n) = \nu_p\left(\binom{n}{m}\right) = \sum_{i=0}^{k} \epsilon_i$ then

$$s_q(m) + s_q(n-m) - s_q(n) = (q-1)\kappa_q(n,m).$$

Define generally $c_q(a_1, a_2, \ldots, a_r)$ as the sum of all carries produced in computing $a_1 + \cdots + a_r$ by the traditional addition algorithm. Then

$$s_q\left(\sum_{i=1}^{r} a_i\right) = \sum_{i=1}^{r} s_q(a_i) - (q-1)c_q(a_1, \ldots, a_r).$$

Schneider, M., Schneider, R.: *Digit sums and generating functions*, Ramanujan J. **52** (2020), 2, 291-302

A factorial excursion

Every positive integer n can be uniquely written in the **factorial base** (**Cantor**) representation

$$n = n_1 \cdot 1! + n_2 \cdot 2! + \cdots + n_k \cdot k! = (n_k \ldots n_2 n_k)_!$$

Ball *et al.* introduce three different analogs of generalized integral binomial coefficients and prove three different analogs, involving generalized factorial base representations, of Kummer's theorem.

Ball, T., Edgar, T., Juda, D.: *Dominance orders, generalized binomial coefficients, and Kummer's theorem*, Math. Mag. **87** (2014), 2, 135-143
Ball, T., Beckford, J., Dalenberg, P., Edgar, T. Rajabi, T.: *Some combinatorics of factorial base representations*, J. Integer Seq. **23** (2020), No.3, Article 20.3.3, 29 p.

Cantor type representation

Let $Q=(q_n)_{n\ge 0}$ be a sequence of positive integers with $q_0=1$ and $q_i>1$ for all $i\ge 1$. Given an $n\in\mathbb{N}$ we have uniquely

$$n=\sum_{j\ge 0} a_{Q,j}(n) q_0\cdots q_j \quad \text{with } 0\le a_{Q,j}(n)<q_{j+1}$$

If $1=q_0<q_1\le q_2\le\ldots$ and $s_Q(n)=\sum_{j=1}^k a_{Q,j}(n)$ then

$$\sum_{n=0}^{m-1} s_Q(n)=\frac{m}{2}\sum_{j=1}^{q^*(m)}(q_j-1)+\frac{mP(m)}{2}+\frac{mq_{q^*(m)}\{P(m)\}^2}{2P(m)}-\frac{m}{2}$$
$$-\frac{mH(P(m))}{P(m)}-\frac{m\{P(m)\}}{2P(m)}+\frac{mq_{q^*(m-1)-1}\{P(m)q_{q^*(m)}\}^2}{2q_{q^*(m)}P(m)}+O\left(\frac{m}{q_{q^*(m)}}\right),$$

where $q^*(m)=i$ denotes the uniquely determined integers ≥ 0 such that $q_i\le m<q_{i+1}$, $P(m)=m/q_{q^*(m)}$, $H(x)=\int_0^x(\{v\}-1)\mathrm{d}v$ and $\{\cdot\}$ is the fractional part.

Kirschenhofer, P., Tichy, R.F.: *On the distribution of digits in Cantor representations of integers*, J. Number Theory **18** (1984), 121-134

Lower bounds for $s_q(n)$

For a wide variety of integer sequences $(a_n)_n$ of controlled growth arising from number theory and combinatorics it was shown that $s_q(a_n)$ is at least $c_q \log n$, where c_q is a constant depending on the base q and on the sequence. For example:

Let $(a_n)_n$ be sequence of positive integers with asymptotic behaviour

$$a_n = e^{f(n)}(1 + O(n^{-\alpha})), \quad \text{with } f'' \asymp \frac{1}{x},$$

for some $\alpha > 0$ and a two times differentiable function f. For any base $q \geq 2$, the inequality

$$s_q(a_n) > \frac{\beta \log n}{10 \log q}, \quad \beta = \min\left\{\alpha, \frac{2}{3}\right\}$$

holds on a set of positive integers n of asymptotic density 1.

Cilleruelo, J., Luca, F., Rue, J., Zumalacárregui, A.: *On the number of nonzero digits of some integer sequences*, Cent. Eur. J. Math. **11** (2012), 188-195

Lower bounds for $s_q(n)$

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for special cases we know improvements

Luca (2002): the number of non-zero digits $l_q(a_n)$ in the base q representation of $a_n = n!$ or $a_n = \mathrm{lcm}[1, 2, \ldots, n]$ grows at least as fast as a constant, depending on base the q, times $\log n$:

$$(l_q(a_n) + 1) \log q + \log(l_q(a_n)) \geq \log(n + 1)$$

Sanna (2015): let $a_n = n!$ or $a_n = \mathrm{lcm}[1, 2, \ldots, n]$ then

$$s_q(n) > C_q \log n \log \log \log n \quad \text{for } n > e^e$$

Luca, F.: *The number of non-zero digits of n!*, Can. Math. Bull. **45** (2002), 1, 115-118
Sanna, C.: *On the sum of digits of the factorial*, J. Number Theory **147** (2015), 836-841

Digits on prescribed positions

Let $C(n,r,d)$ be the number of 1 bits in the binary representation of n that are in positions that are congruent to $r \pmod{d}$, the positions are indexed starting at 0 on the right. Then

$$\sum_{n\geq 0} C(n,r,d)z^n = \sum_{m\geq 0} \frac{z^{2^{r+dm}}}{1+z^{2^{r+dm}}}, \quad d \geq 0, 0 \leq r < d$$

For instance, the generating function for the number of 1's in even positions in the binary expansion of n is given by

$$\frac{1}{1-z}\sum_{m=0}^{\infty} \frac{z^{4^m}}{1+z^{4^m}}$$

Adams-Watters, F.T., Ruskey, F.: *Generating functions for the digital sum and other digit counting sequences*, J. Integer Seq. **12** (2009), N0. 5, Article ID 09.5.6, 9 p.

Power sums of digital sums

Motivation: Glaisher (§14) has shown that the number of odd binomial coefficients $\binom{n}{j}$, where $0 \le j \le n$, is $2^{s_2(n)}$. Consequently, the number of odd numbers in the first k rows of Pascal's triangle is $\sum_{n=0}^{k-1} 2^{s_2(n)} = \sum_{j=0}^{\infty} \left(\sum_{n=0}^{k-1} s_2(n)^k\right) \frac{(\log 2)^j}{j!}$.

Stolarsky:

$$\frac{1}{x} \sum_{n<x} s_2(n)^k = \left(\frac{\log x}{\log 2}\right)^k + \begin{cases} O\left((\log x)^{k-1}\right), & k \text{ non-negative integer} \\ O\left((\log x)^{k-1/2}\sqrt{\log \log x}\right), & k \ge 0. \end{cases}$$

Coquet extended the first estimate to arbitrary real k proving:

$$\sum_{n<N} s_2(n)^k =$$
$$N\left(\frac{\log N}{2\log 2}\right)^k + N\left(\frac{\log N}{2\log 2}\right)^{k-1}\left(kF\left(\frac{\log N}{\log 2}\right) + \frac{k(k-1)}{4}\right) + O\left(N\left(\frac{\log N}{\log 2}\right)^{k-2}\right)$$

with a function $F : \mathbb{R} \to \mathbb{R}$ of period 1, continuous, nowhere differentiable, and the implicit constant depending only on k.

Coquet, J.: *Power sums of digital sums*, J. Number Theory **22** (1986), 161-176
Glaisher, J. W. L.: *On the residue of a binomial-theorem coefficient with respect to a prime modulus*, Quart. J. **30** (1899), 150-156
Stolarsky, K.B.: *Power and exponential sums of digital sums related to binomial coefficient parity*, SIAM J. Appl. Math. **32** (1977), 717-730

Digit sums over prime bases

$s_q(n)$ is, on average, not too dependent on the primality of q Fissum:

$$\sum_{\substack{p \leq N \\ p \text{ prime}}} s_p(n) = \left(1 - \frac{\pi^2}{12}\right) \frac{N^2}{\log N} + C \frac{N^2}{\log^2 N} + o\left(\frac{N^2}{\log^2 N}\right), \qquad C \doteq 0.1199$$

Fissum, R.: *Digit sums and the number of prime factors of the factorial* $n! = 1 \cdot 2 \cdot n$, (Bachelor's project in BMAT), Norwegian University of Science and Technology, Tromdheim, Gjøvik May 2020 (Prop. 2.12).

Digit sums over complex primes

Mauduit & Rivat answering a question by Gel'fond proved that the sum of digits of prime numbers written in a basis $q \geq 2$ is equidistributed in arithmetic progressions (except for some well known degenerate cases): if $q \geq 2$ and $m \geq 2$ and there exists $\sigma_{q,m}$ such that for every $a \in \mathbb{Z}$ we have

$$\mathrm{card}\{p \leq x : s_q(p) \equiv a \pmod{m}\} = \frac{(m, q-1)}{m}\pi(x, a, (m, q-1)) + O(x^{1-\sigma_{q,m}})$$

Drmota, Rivat & Stoll extended this result to Gaussian primes from some fixed residue class lying in full disc and basis $-a \pm \mathrm{i}$.

Morgenbesser extended further this result to Gaussian primes from some fixed residue class lying in angular regions and basis $-a \pm \mathrm{i}$.

Gelfond, A.O.: *Sur les nombres qui ont des propriétés additives et multiplicatives données*, Acta Arith. **13** (1968), 259-265
Drmota, M., Rivat, J., Stoll, Th.: *The sum of digits of primes in $\mathbb{Z}[i]$*, Monatsh. Math. **155** (2008), 3-4, 317-347
Morgenbesser, J.F.: *The sum of digits of Gaussian primes*, Ramanujan J. **27** (2012), 1, 43-70
Mauduit, Ch., Rivat, J.: *Sur un problème de Gelfond: la somme des chiffres des nombres premiers*, Ann. Math. **171** (2010), 1591-1646

Functional generalisations

Bellman & Shapiro in connection with function $s_2(n)$ proposed to investigate arithmetic functions $w(n)$, called by them **dyadically additive**, satisfying the relation $w(m+n) = w(m) + w(n)$, whenever m and n have no summand in common when written as sums of distinct powers of two.

Bellman, R., Shapiro, H.N.: *On a problem in additive number theory*, Ann. Math. (2) **49** (1948), 333-340
Gelfond, A.O.: *Sur les nombres qui ont des propriétés additives et multiplicatives données*, Acta Arith. **13** (1968), 259-265
Mahler, K.: *The spectrum of an array and its application to the study of the translation properties of a simple class of arithmetical functions. II: On the translation properties of a simple class of arithmetical functions*, J. Math. Phys., Mass. Inst. Techn. **6** (1927), 158-163 (Reprinted in Publ. MIT Ser. II 62 No. 118 (1927)).
Mendès France, M.: *Les suites ŕ spectre vide et la répartition modulo 1*, J. Number Theory **5** (1973), 1-15

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During his visit to Paris in October 1966 (cf. Mendès France), A.O.Gel'fond defined the notion of the q-additive function: Given an integer $q \geq 2$, an arithmetic function $f : \mathbb{N}_0 \to \mathbb{R}$ is called q-**additive** if for every $r, a \in \mathbb{N}$ and $b \in \mathbb{N}_0$ we have $f(q^r a + b) = f(q^r a) + f(b)$ whenever $0 \leq b < q^r$.

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In the written form, in his last but one research paper (Gelfond) he only defined the notion of the **additive function** as a function satisfying the relations $f(n) = f(a) + f(b)$ if $n = a + b$ and $a < 2^\ell$, $b = 2^\ell c$, where $a, b, c, n \in \mathbb{N}$.

As examples of such additive functions he gives the identity function, the sum-of-digits functions s_q and their linear combinations.

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Traces of the concept of the q-additive function can also be found in (Mahler).

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A later extension of this definition says that a function $f : \mathbb{N}_0 \to \mathbb{R}$ is said to be **strongly** q-**additive** if $f(q^r a + b) = f(a) + f(b)$ whenever $0 \leq b < q^r$.

Bellman, R., Shapiro, H.N.: *On a problem in additive number theory*, Ann. Math. (2) **49** (1948), 333-340
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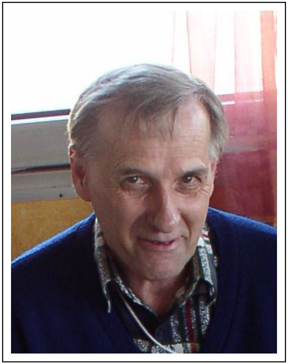
Joint work with

L.Mišík, Ostrava



O.Strauch, Bratislava

Mišík, L., Porubský, Š., Strauch, O.: *Uniform distribution of the weighted sum-of-digits functions*, Unif. Distrib. Theory **16** (2021), no. 1, 93-126

Uniform distribution mod 1

Sequence of d-dimensional vectors $\vec{x}_n$, $n = 0, 1, 2, \ldots,$ in $\mathbb{R}^d$ is said to be **uniformly distributed mod 1** (shortly u.d. mod1) if for all intervals $[\vec{a}, \vec{b}) \subseteq [0, 1)^d$ we have

$$\lim_{N\to\infty} \frac{A\left([\vec{a}, \vec{b}); N; \vec{x}_n \bmod 1\right)}{N} = \prod_{j=1}^{d} (b_j - a_j)$$

where $\vec{a} = (a_1, \ldots, a_d)$ and $\vec{b} = (b_1, \ldots, b_d)$.

Here, $A(I; N; \vec{x}_n)$ denotes the number of elements, out of the first N elements of the sequence $\vec{x}_n$, $n = 0, 1, 2, \ldots,$ that lies in set $I \subseteq \mathbb{R}^d$.

Almost uniform distribution mod 1

If there exists an increasing sequence of positive integers $\mathfrak{N} = \{N_1 < N_1 < N_3 < \dots\}$ such that

$$\lim_{j\to\infty} \frac{A\left([\vec{0}, \vec{x}); N_j; \vec{x}_n \bmod 1\right)}{N_j} = \prod_{j=1}^{d} x^{(j)} \tag{1}$$

for all $\vec{x} \in [0,1)^d$ then the sequence $\vec{x}_n$, $n = 0, 1, 2, \dots$, is called **$\mathfrak{N}$-almost uniformly distributed mod1** (or $\mathfrak{N}$-almost u. d. mod1).

ALMOST UNIFORM DISTRIBUTION $\bmod 1$

If there exists an increasing sequence of positive integers
$\mathfrak{N} = \{N_1 < N_1 < N_3 < \dots\}$ such that

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$\frac{n}{2^{1+\lfloor \log_2 n \rfloor}}$ — almost u.d. but not u.d.

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$\frac{n}{2^{1+\lfloor \log_2 n\rfloor}}$ — almost u.d. but not u.d.

$\log p_n$, p_n the nth prime — not almost u.d.

S. Akiyama: *Almost uniform distribution modulo 1 and the distribution of primes*, Acta Math. Hung. **78** (1998), 39-44

Let d be a positive integer and let $s_q^{(d)}(n) = \sum_{j=0}^{\infty} n_j^d$ denote the sum of the dth powers of the q-adic digits of the positive integer n. If $\theta \in \mathbb{R}$ then sequences of the form $\theta s_q^{(d)}(n)$ with n running over $\mathbb{N}_0$ or over the set of prime numbers were studied by several authors.

s_q AND U.D.

MENDÈS FRANCE proved that sequence $\theta s_q(n)$, $n = 0, 1, 2, \dots,$ is u.d. mod1 if and only if θ is irrational.

This result was later reproved by COQUET, who proved that for every $k \in \mathbb{N}$ the sequence $\theta s_q^{(k)}(n)$, $n = 0, 1, 2, \dots,$ is u.d. mod 1 if and only if θ is irrational.

MAUDUIT & RIVAT proved that $\theta s_q^{(1)}(n)$ is u.d. mod 1 when θ is irrational and n runs through the prime numbers only.

TICHY & TURNWALD proved estimates for the discrepancy of the sequence $\alpha s_q^{(d)}(n)$, $n = 0, 1, 2, \dots,$ for irrational α of finite approximation type η.

DRMOTA & RIVAT & STOLL proved that sequence $(\alpha s_{-a\pm\mathrm{i}}(p))$, running over Gaussian primes p is uniformly distributed modulo 1 if and only if $\alpha \in \mathbb{R}/\mathbb{Q}$ if $-a \pm \mathrm{i}$ is prime and $a \geq 28$. MORGENBESSER extended this result to circular sector. Simultaneously he removed the conditional assuptions.

M. MENDÈS FRANCE: *Nombres normaux applications aux fonctions pseudoaleatoires*, J. Anal. Math. **20** (1967), 1-56
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WEIGHTED q-ARY SUM-OF-DIGITS FUNCTION

Let

$$\boldsymbol{\gamma} = (\gamma_0, \gamma_1, \gamma_3, \dots)$$

denote a sequence of real numbers, $q \geq 2$ a positive integer, and $\Sigma_q = \{0, 1, \dots, q-1\}$ the set of q-ary digits.

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$$n = n_0 + n_1 q + n_2 q^2 + \cdots + n_\ell q^\ell, \quad n_j \in \Sigma_q, n_\ell \neq 0,$$

where $\ell = \lfloor \log_q n \rfloor$,

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q-adic van der Corput sequence $= s_{q,\gamma}(n)$, $n \in \mathbb{N}_0$, where $\gamma_i = q^{-i-1}$ for all $i \in \mathbb{N}_0$.

For a non-negative integer n with base q, $q \geq 2, q \in \mathbb{N}$, representation

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$$\phi_q(n) = \frac{n_0}{q} + \frac{n_1}{q^2} + \frac{n_2}{q^3} + \ldots \frac{n_k}{q^{k+1}}.$$

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van der Corput sequence

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$$\phi_q(n), \quad n = 0, 1, 2 \ldots,$$

is an example of the simplest one-dimensional low-discrepancy sequence over the unit interval. Simultaneously, it can be interpreted as an example of a weighted q-ary sum-of-digits function.

Halton sequence

If q_i, $i \in \{1, \ldots, d\}$, are pairwise coprime bases, the d-**dimensional Halton sequence** is defined by

$$(\phi_{q_1}(n), \phi_{q_2}(n), \ldots, \phi_{q_d}(n)), \quad n = 0, 1, 2 \ldots .$$

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$d = 1 \looparrowright$ van der Corput

d-DIMENSIONAL GENERALIZATION WITH $d > 1$

Let $(q_1, q_2, \ldots, q_d)$ be a d-tuple of positive integers ≥ 2 and

$$\Gamma = \begin{pmatrix} \boldsymbol{\gamma}^{(1)} \\ \boldsymbol{\gamma}^{(2)} \\ \vdots \\ \boldsymbol{\gamma}^{(d)} \end{pmatrix} = \begin{pmatrix} \gamma_0^{(1)} & \gamma_1^{(1)} & \gamma_2^{(1)} & \cdots \\ \gamma_0^{(2)} & \gamma_1^{(2)} & \gamma_2^{(2)} & \cdots \\ \vdots & \vdots & \vdots & \ddots \\ \gamma_0^{(d)} & \gamma_1^{(d)} & \gamma_2^{(d)} & \cdots \end{pmatrix} = (\vec{\gamma}_0^T, \vec{\gamma}_1^T, \vec{\gamma}_2^T, \ldots)$$

be a $d \times \infty$-matrix with real entries with $\vec{\gamma}_j = (\gamma_j^{(1)}, \gamma_j^{(2)}, \ldots, \gamma_j^{(d)})$ transposed in the jth column.

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be a $d \times \infty$-matrix with real entries with $\vec{\gamma}_j = (\gamma_j^{(1)}, \gamma_j^{(2)}, \dots, \gamma_j^{(d)})$ transposed in the jth column. For every $n \in \mathbb{N}_0$ define

$$s_{q_1,\dots,q_d,\Gamma}(n) = \left(s_{q_1,\boldsymbol{\gamma}^{(1)}}(n), s_{q_2,\boldsymbol{\gamma}^{(2)}}(n), \dots, s_{q_d,\boldsymbol{\gamma}^{(d)}}(n)\right),$$

Pillichshammer's problem

Pillichshammer proposed the following general problem:

Let $q_1, \ldots, q_d$ be a d-tuple of pairwise coprime integers ≥ 2. What properties of the weight sequences forming Γ guarantee the uniform distribution mod 1 of the sequence

$$s_{q_1,\ldots,q_d,\Gamma}(n) = \left(s_{q_1,\gamma^{(1)}}(n), s_{q_2,\gamma^{(2)}}(n), \ldots, s_{q_d,\gamma^{(d)}}(n)\right), \qquad n = 0, 1, 2, \ldots ?$$

F. Pillichshammer: *Uniform distribution of sequences connected with the weighted sum-of-digits function*, Unif. Distrib. Theory **2** (2007), 1, 1-10

Partial answer

Pillichshammer proved:

Let the base $q \in \mathbb{N}$ be at least 2. The sequence

$$s_{q,\Gamma}(n) = \left(s_{q,\gamma^{(1)}}(n), s_{q,\gamma^{(2)}}(n), \ldots, s_{q,\gamma^{(d)}}(n)\right)$$

is u.d. mod1 if and only if for every integral vector $\vec{h} \in \mathbb{Z}^d \setminus \{\vec{0}\}$ one of the following conditions is fulfilled: either

$$\sum_{\substack{k=0 \\ \langle \vec{h}, \vec{\gamma}_k \rangle q \notin \mathbb{Z}}}^{\infty} ||\langle \vec{h}, \vec{\gamma}_k \rangle||^2 = \infty,$$

or, there exists a non-negative integer k with

$$\langle \vec{h}, \vec{\gamma}_k \rangle \notin \mathbb{Z} \quad \text{and} \quad \langle \vec{h}, \vec{\gamma}_k \rangle q \in \mathbb{Z}.$$

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van der Corput sequence satisfies the second condition of this criterion with k taken as the maximal exponent such that q^{k+1} divides h.

HOFER proved a sufficient condition on the weight sequences which gives a partial answer to PILLICHSHAMMER's question which requires the divergence of the series

$$\sum_{i=0}^{\infty} \left\| h\left(\gamma_{2i+1}^{(j)} - q_j \gamma_{2i}^{(i)}\right) \right\|^2$$

for each dimension $j \in \{1, \ldots, d\}$ and every non-zero integer h.

Drawback: this sufficient condition is not necessary and it does not cover some prototype classes of u.d. sequences as d-dimensional Kronecker sequences.

R.HOFER: *Note on the joint distribution of the weighted sum-of-digits function modulo one in case of pairwise coprime bases*, Unif. Distrib. Theory **2** (2007), 2, 35-47

Trigonometric criterion

Let $q \geq 2$ be an integer and Γ be the $d \times \infty$-matrix of real weights defined above. Then the sequence $s_{q,\Gamma}(n)$, $n = 0, 1, 2, \ldots$, is u.d. mod 1 if and only if for every integral vector $\vec{h} \in \mathbb{Z}^d \setminus \{\vec{0}\}$ we have

$$\lim_{N\to\infty} \prod_{\substack{j=0 \\ \langle \vec{h}, \vec{\gamma}_j \rangle \notin \mathbb{Z}}}^{N-1} \frac{\sin \pi ||q\langle \vec{h}, \vec{\gamma}_j \rangle||}{q \sin \pi ||\langle \vec{h}, \vec{\gamma}_j \rangle||} = 0. \qquad (P)$$

Mišík, L., Porubský, Š., Strauch, O.: *Uniform distribution of the weighted sum-of-digits functions*, Unif. Distrib. Theory **16** (2021), no. 1, 93-126

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A DISCREPANCY ESTIMATE

Let $q \geq 2, N, M$ be positive integers such that $q^N \leq M < q^{N+1}$. Let Γ be the $d \times \infty$-matrix of real weights as above. Then for the discrepancy of the sequence

$$s_{q,\Gamma}(n) \mod 1, \quad n = 0, 1, 2, \ldots, M-1,$$

we have

$$D_M(s_{q,\Gamma}(n) \bmod 1) \leq C_d \left(\frac{1}{H} + \sum_{0 < ||\vec{h}||_\infty \leq H} \frac{1}{r(\vec{h})} \times \right.$$
$$\left. \times \left(\sum_{j=1}^{k} q^{-j+2} \prod_{\substack{t=0 \\ \langle \vec{h}, \vec{\gamma}_t \rangle \notin \mathbb{Z}}}^{N-j} \frac{|\sin \pi q \langle \vec{h}, \vec{\gamma}_t \rangle|}{q |\sin \pi \langle \vec{h}, \vec{\gamma}_t \rangle|} + O\left(\frac{1}{q^{k-1}} \right) \right) \right).$$

for every integer k satisfying $1 \leq k \leq N$.

DISTRIBUTION FUNCTIONS OF $s_{q,\gamma}(n)$ mod 1

Distribution function $g(x)$ is called a **distribution function of sequence** x_n, $n = 1, 2, \dots,$ if there exists an increasing sequence of positive integers $N_1, N_2, \dots$ such that

$$\lim_{k\to\infty} \frac{A\big([0,x), N_k, x_n\big)}{N_k} = g(x) \text{ a.e. on } [0,1).$$

Distribution function $g(x)$ is called an **asymptotic distribution function** of the sequence x_n, $n = 1, 2, \dots,$ if $\lim_{N\to\infty} F_N(x) = g(x)$ a.e. on $[0,1]$. Sequence x_n, $n = 1, 2, \dots,$ is u.d. in $[0,1]$ if and only if $g(x) = x$ is its asymptotic distribution function.

There holds:

If function $g(x) = x$, $x \in [0,1]$, is a distribution function of the sequence $s_{q,\gamma}(n)$ mod 1, $n = 0, 1, 2, \dots,$ then this sequence is u.d., i.e. $g(x) = x$ is its asymptotic d.f.

A new property of the van der Corput sequence

$$(q-1)\sum_{j=0}^{\infty}\gamma_j = S \tag{S1}$$

$$\gamma_0 \geq \gamma_1 \geq \gamma_2 \geq \cdots > 0. \tag{S2}$$

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If there exists $\lambda = 0, 1, 2, \ldots$ such that

$$(q-1)(\gamma_{\lambda+2} + \gamma_{\lambda+3} + \ldots) < \gamma_{\lambda+1}.$$

Then the interval

$$J = \left((q-1)\sum_{j=0}^{\infty}\gamma_j - \gamma_{\lambda+1}, (q-1)\sum_{j=0}^{\lambda+1}\gamma_j\right)$$

(of positive length) does not contain an element of the form $s_{q,\gamma}(n)$.

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If there exists $\lambda = 0, 1, 2, \ldots$ such that

$$(q-1)(\gamma_{\lambda+2} + \gamma_{\lambda+3} + \ldots) > \gamma_{\lambda+1},$$

then sequence $s_{q,\gamma}(n)$, $n = 0, 1, 2, \ldots,$ is not u.d. in the interval $[0, S]$.

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$$(q-1)\sum_{j=0}^{\infty}\gamma_j = S \tag{S1}$$

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Let γ be a sequence of positive real numbers such that for every $\lambda = 0, 1, 2, \ldots$ we have

$$(q-1)(\gamma_{\lambda+2} + \gamma_{\lambda+3} + \ldots) = \gamma_{\lambda+1}.$$

If γ satisfies conditions (S1) and (S2), and $S = 1$ then sequence $s_{q,\gamma}(n)$, $n = 0, 1, 2, \ldots,$ is the q-adic van der Corput sequence.

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If γ satisfies conditions (S1) and (S2), and $S = 1$ then sequence $s_{q,\gamma}(n)$, $n = 0, 1, 2, \ldots,$ is the q-adic van der Corput sequence.

Consequently, if γ satisfies conditions (S1), (S2), and $S = 1$ then every uniformly distributed γ-weighted q-adic sum-of-digits function $s_{q,\gamma}(n)$, $n = 0, 1, 2, \ldots,$ is the q-adic van der Corput sequence.

Thank you for your kind attention.