Improved bounds on some S-unit equations (joint work with Kálmán Győry)

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Plan of the talk

- General principle of the method
- Improved bounds on the S-unit equation.
- Applications
- The case $x^n + y = 1$, and perspectives

Fixed notation and assumptions

- K is a number field, M_K its set of places (prime ideals + archimedean norms).
- $S \subset M_K$ is finite, contains the infinite places, so that

$$\mathcal{O}_{K,S} := \{x \in K \,|\, \forall v \in M_K \setminus S,\ |x|_v \leq 1\}.$$

- C is a projective smooth irreducible curve over K.
- D is an effective divisor on C.
- The *S-integral points of $C \setminus D$* are then defined by

$$(C \backslash D)(\mathcal{O}_{K,S}) := \{P \in C(K) \,|\, \forall \mathfrak{p} \in M_K \backslash S,\ (P \mod \mathfrak{p}) \notin D \mod \mathfrak{p}\}.$$

Example

$$(\mathbb{P}^1 \setminus \{0, \infty\})(\mathcal{O}_{\mathbb{Q},\{\infty,2,3\}}) = \mathcal{O}_{\mathbb{Q},\{\infty,2,3\}}^{\times} = \{\pm 2^a 3^b, a, b \in \mathbb{Z}\}.$$

Effective bounds on integral points

Theorem (Siegel-Faltings)

If $g(C) + 2|D| \geq 3$, $(C\backslash D)(\mathcal{O}_{K,S})$ *is finite for all* S.

Problem

This result does not tell us at all how to find the integral points.

Definition (Weil height)

The Weil height of $x \in K^*$ is defined by

$$h(x) = \frac{1}{[K:\mathbb{Q}]} \sum_{v \in M_K} n_v \log^+(|x|_v),$$

with $n_v = [K_v : \mathbb{Q}_{v_0}]$ for $v|v_0 \in M_\mathbb{Q}$ the local degree of v.

Goal of an effective bound

For $\phi \in K(C)$ fixed with poles in D, find a bound on the height $h_\phi := h \circ \phi$ on integral points of $(C \setminus D)$.

The general idea for the method

In everything that follows:

- $d = [K : \mathbb{Q}]$
- $s = |S|$
- For $v \in M_K$, $N(v) = N(\mathfrak{p}) = |\mathcal{O}_K/\mathfrak{p}|$ if $v = \mathfrak{p}$ prime ideal, 1 if v archimedean.
- For $r \in \mathbb{N}^*$, $N^{(r)}(S)$ is the r-th largest $N(v)$, $v \in S$.
- R_S = regulator of $\mathcal{O}_{K,S}$, $\log^*(x) = \max(\log(x), 1)$.

Theorem (LF, 2019)

Assume (B): "for every $Q \in D$, there is $\phi_Q \in K(C)$ nonconstant with poles and zeroes in $D \setminus \{Q\}$ and $\phi_Q(Q) = 1$".
Then, there is $c > 0$ effective absolute such that for all S,

$$h_\phi(P) \leq c f(d,s) \frac{N^{(r)}(S)}{\log^* N^{(r)}(S)} h_K R_S \log^*(h_K R_S)$$

where r is the number of $\mathrm{Gal}(\overline{K}/K)$-orbits of D and $f(d,s)$ explicit.

Idea of proof

$$D = \bigsqcup_{i=1}^{r} D_i, \quad P \in (C \setminus D)(\mathcal{O}_{K,S}).$$

Definition (Local height)

For $1 \leq i \leq r$ and $v \in M_K$, define the local and global heights

$$\begin{aligned} h_{D_i,v}(P) &\simeq \max(-\log(v\text{-adic distance between } P \text{ and } D_i), 0) \\ h_{D_i}(P) &= \frac{1}{[K:\mathbb{Q}]} \sum_{v \in M_K} n_v h_{D_i,v}(P), \end{aligned}$$

(notice $h_{D_i,v}(P) > 0$ if and only if $P \mod v \in D_i$).

Idea of Proof, part I (goal: bound some height(s)).

- For each $v \in S$, $h_{D_i,v}(P)$ large at most for one i.
- Take $S' = S \setminus \{(r-1) \text{ largest } v\text{'s}\}$.
- Pigeonhole principle : $\exists i$, $h_{D_i,v}(P)$ small for all $v \in S \setminus S'$, keep it.
- There is $v \in S'$ such that $h_{D_i,v}(P) \gtrsim 1/|S'| h_{D_i}(P)$, keep it.
- Enough to bound $h_{D_i,v}(P)$ then!

□

Idea of proof, the end

By (B), one has $\phi_i \in K(C)$ such that $\phi_i(D_i) = 1$ and $\mathrm{div}(\phi_i) \subset D$ and then $\phi_i(P) \in \mathcal{O}_{K,S}^\times$ (almost), $h_{1,v}(\phi_i(P)) \gtrsim h_{D_i,v}(P)$ and $h_{D_i}(P) \gg h(\phi_i(P))$, so we have

$$v \in S', \text{ and } h_{1,v}(\phi_i(P)) \gtrsim h_{D_i,v}(P) \gtrsim \frac{h_{D_i}(P)}{|S'|} \gg \frac{h(\phi_i(P))}{|S'|}.$$

Proposition (Evertse and Győry, 2015)

Let Γ subgroup of K^* generated by $\{\xi_1, \cdots, \xi_{s-1}\}$ up to torsion. $\Theta_\Gamma := h(\xi_1) \cdots h(\xi_{s-1})$. Then, for every $\xi \neq 1 \in \Gamma$ and every $v \in K$,

$$h_{1,v}(\xi) = -\log|1-\xi|_v < (16ed)^s \frac{N(v)}{\log^* N(v)} \Theta_\Gamma \log^*(N(v)h(\xi)).$$

Consequence

All the previous work allows to apply this to $\xi = \phi_i(P), \Gamma = \mathcal{O}_{K,S}^*$ and with $v \in S'$ so $N(v) \leq N^{(r)}(S)$ and Θ_Γ bounded in terms of R_S (see later),we thus obtain the theorem.

Applications to the S-unit equations: previous bounds

With fixed $\alpha, \beta \in K$, consider

$$\alpha x + \beta y = 1, \quad x, y \in \mathcal{O}_{K,S}^{\times}. \tag{1}$$

and define $H := \max(h(\alpha), h(\beta), 1)$.

Remark

For $\alpha = \beta = 1$, these points correspond to $(\mathbb{P}^1 \setminus \{0, 1, \infty\})(\mathcal{O}_{K,S})$: $r = 3$, (B) verified with functions $1 - x, x, 1 - 1/x$.

Lemma (Hajdu, 1993)

There exists a fundamental system of S-units $\xi_1, \cdots \xi_{s-1}$ for $\Gamma = \mathcal{O}_{K,S}^{\times}$ such that $\Theta_\Gamma \leq s^{2s} R_S$.

Theorem (Győry-Yu, 2006)

All solutions of (1) *satisfy*

$$\max(h(x), h(y)) \leq AH \tag{2}$$

where

$$A = (16ds)^{2s+3} N^{(1)}(S) \left(1 + \frac{\log^* R_S}{\log^* N^{(1)}(S)}\right) R_S.$$

Improvements on the bounds

Theorem (LF, 2020)

With (1) *and in* (2), *for* $P' = N^{(3)}(S)$ *one can take*

$$A = 2 \cdot (16ds)^{2s+3} P' \left(1 + \frac{\log^* R_S}{\log^* P'}\right) R_S.$$

Remark

The only dependance in the largest place v of S is in R_S (thus in $\log N(v)$).

Theorem (Győry, 2020)

t *number of finite places in* S, $\mathcal{R} = \max(h_K, R_K)$. *With* (1) *and in* (2), *one can take*

$$A = (16ed)^{4s} \mathcal{R}^{t+4} \frac{P'}{\log^* P'} \left(1 + \frac{\log^* \log P}{\log^* P'}\right) R_S.$$

Remark

Big improvement on the blue factor : no more s^s term, but slightly weaker dependence in R_K.

Applications to explicit abc bounds over number fields

For $a, b, c \in K^*$ with $a + b + c = 0$, define

$$H_K(a,b,c) := \prod_{v \in M_K} \max(\|a\|_v, \|b\|_v, \|c\|_v) \quad N_K(a,b,c) := \prod_{\mathfrak{p}} N(\mathfrak{p})^{e(\mathfrak{p}/p)}$$

where $\mathfrak{p}$ goes through primes for which we do <u>not</u> have $|a|_\mathfrak{p} = |b|_\mathfrak{p} = |c|_\mathfrak{p}$.

Uniform abc conjecture over number fields

$$H_K(a,b,c) \ll_{\varepsilon,[K:\mathbb{Q}]} (\Delta_K N_K(a,b,c))^{1+\varepsilon}$$

Theorem (Győry, 2022)

For all a, b, c as above and $N = \max(N_K(a,b,c), 16)$,

$$\log H_K(a,b,c) < c_1 N^{1/3 + \frac{c_2 \log_3 N}{\log_2 N}}$$

with c_1, c_2 eff. computable in d, Δ_K and $\log_r(x) = \log \circ \cdots \circ \log x$.

Remark

Previous exponent of $1 + \varepsilon$ given by Győry in 2008.

New applications

For $\Gamma \subset K^*$ a finitely generated subgroup of rank m, define the division group

$$\overline{\Gamma} := \{x \in \overline{\mathbb{Q}}^* \mid \exists k \in \mathbb{Z}_{>0},\, x^k \in \Gamma\}.$$

Consider for fixed $\alpha, \beta \in \overline{\mathbb{Q}}^*$ (again, $H = \max(h(\alpha), h(\beta), 1)$) the equation

$$\alpha x + \beta y = 1, \qquad x, y \in \overline{\Gamma}. \tag{3}$$

Theorem (Győry-LF, 2022)

For solutions of (3) *with* $\Gamma \subset \mathcal{O}_{K,S}^{\times}$, $[K(\alpha,\beta,x,y) : K(\alpha,\beta)] \leq 2$ *and*

$$\max(h(x), h(y)) \leq A(H + mh_0) + mh_0$$

with

$$\begin{aligned} h_0 &= \max_{1 \leq i \leq m} h(\xi_i) \\ A &= 16 c_6 s \frac{P'}{\log^* P'} \Theta_\Gamma \max(\log(c_6 s P'), \log^* \Theta_\Gamma), \end{aligned}$$

c_6 *explicit in* d *and* m *and again* $P' = N^{(3)}(S)$.

Other new applications

Other results for solutions of $\alpha x + \beta y = 1$ for:

- $(x, y) \in \mathbf{\Gamma}$ a finite type subgroup of $(\overline{\mathbb{Q}}^*)^2$ generated up to torsion by $\underline{\xi_1}, \cdots, \underline{\xi_m}$, and such that $\mathbf{\Gamma} \subset (\mathcal{O}_{K,S}^{\times})^2$.

Theorem (Győry-LF, 2022)

We again have $[K(x, y, \alpha, \beta) : K(\alpha, \beta)] \leq 2$ *and*

$$\max(h(x), h(y)) \leq A(H + 3mh_0),$$

with $h_0 = \max_i(h(\underline{\xi_i}))$, $A = 16c_6' s \frac{P'}{\log^* P'} \Theta_{\mathbf{\Gamma}}, \max(\log(c_6' s P'), \log^* P')$, $c_6' = m^2(16ed)^{3(m+2)}$.

- Same bound for $\overline{\mathbf{\Gamma}}_\varepsilon := \{\underline{x} \in (\overline{\mathbb{Q}}^*)^2 \,|\, \underline{x} = \underline{y}\underline{z} \text{ with } \underline{y} \in \mathbf{\Gamma}, h(\underline{z}) < \varepsilon\}$.
- Similar bound for $C(\overline{\mathbf{\Gamma}}, \varepsilon) := \{\underline{x} \in (\overline{\mathbb{Q}}^*)^2 \,|\, \underline{x} = \underline{y}\underline{z} \text{ with } \underline{y} \in \mathbf{\Gamma}, h(\underline{z}) < \varepsilon(1 + h(\underline{y})\}$.

The "power" unit equation $x^n + y = 1$

Start now with the equation

$$x^n + y = 1, \quad (x, y) \in \mathcal{O}_{K,S}^{\times} \tag{4}$$

Starting point

Solutions correspond to points of $(\mathbb{P}^1 \setminus D)(\mathcal{O}_{K,S})$, where

$$D = \{0, \infty\} \cup \mu_n.$$

This gives:

- $r = \tau(n) + 2$ Galois orbits over $\mathbb{Q}$ as $\mu_n = \sqcup_{d|n} \mu_d^*$.
- Hypothesis (B) ("for every $Q \in D$, there is $\phi_Q \in K(C)$ nonconstant with poles and zeroes in $D \setminus \{Q\}$") satisfied with:
  - $1 - x$ for 0.
  - $1 - 1/x$ for ∞.
  - x^n for any n-th root of unity.

Shape of the result

$$x^n + y = 1$$

Theorem (Győry-LF, 2022)

For the solutions of (4), *and* $P' = N^{(\tau(n)+2)}(S)$, *we have a bound of the shape*

$$\max\left(h(x), \frac{h(y)}{n}\right) \leq C(d,s)\frac{P'}{\log^* P'}\left(1 + \frac{\log^* R_S}{\log^* P'}\right) R_S$$

with $C(d,s)$ *explicit in* d, s.

Consequence

When n has a lot of divisors, the red factor can be much smaller than the one for $\alpha x + \beta y = 1$.

Remark

If $s < \tau(n) + 2$, we have the much better bound $2\log(2n)$ by Runge's method.

Perspectives (work in progress)

Could we do the same for the more general equation

$$\alpha x^n + \beta y = 1, \quad (x, y) \in \mathcal{O}_{K,S}^\times \quad ?$$

Problem

In this case, the solutions correspond to points of $(\mathbb{P}^1 \setminus D)(\mathcal{O}_{K,S})$ with

$$D = \{0, \infty\} \cup \{n\text{-th roots of } 1/\alpha\}.$$

In general, these n-th roots are all conjugate over $\mathbb{Q}$!

Possible strategy

Extend the field K to have slightly more Galois orbits ?

Thank you for your attention!