$$(x^2+1)(y^2+1)=z^2+1$$

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Motivation

Vica is 4
Samu is 9
Their mother Kata is 36

$$4 \cdot 9 = 36,\ \ 4 + 9 + 36 = 49.$$

Any more such triples?

The equation

is now

$$x^2 + y^2 + (xy)^2 = z^2$$

By adding 1 to both sides we get the title equation

$$(x^2 + 1)(y^2 + 1) = z^2 + 1$$

which is beautiful but does not help in the solution.

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The positive setting

The equation is symmetirc in x, y and everything is squared: we may assume $0 \leq x \leq y$.
Exclude the trivial solution $(0, 0, 0)$. This is the only one with $x = y$, so we may assume $0 \leq x < y$.
If $x = 0$, then $y = z$; we call this class $(0, n, n)$ of solutions the *root*.

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The positive recursion

Theorem

All solutions can be obtained from the root by repeated application of the transformations

$$(T1)\qquad \begin{cases} x' = y, \ \ y' = 2y(z - xy) - x \\ z' = z + 2y\big(y(z - xy) - x\big), \end{cases}$$

$$(T2)\qquad \begin{cases} x' = y, \ \ y' = 2y(z + xy) + x, \\ z' = z + 2y\big(y(z + xy) + x\big), \end{cases}$$

$$(T3)\qquad \begin{cases} x' = x, \ \ y' = 2x(z + xy) + y, \\ z' = z + 2x\big(x(z + xy) + y\big). \end{cases}$$

Comment

A root solution $(0, n, n)$ is a fixed point of (T3), while (T1) and (T2) both turn it into $(n, 2n^2, 2n^3 + n)$. From this on different sequences of transf ormations yield different soltuions, that is, **these transformations act as a free semigroup**.

Call the next level $(n, 2n^2, 2n^3 + n)$ the *stem*.
Each root gives rise to a single stem, from which grow 3 branches, which again ramify in 3 directions, up to the sky.

Another remarkable set is the sequence $(n, n + 1, n^2 + n + 1)$, obtained from $(1, 2, 3)$ by a repeated application of (T1). We call this the *main sequence*, as this gives the majority of solutions (more on this later).

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Highlights from the proof

The ransformations (T1)–(T3) climb up in the tree of solutions. Now we shall climb down.
The substitution $z = xy + t$ turns the equation into

$$x^2 + y^2 - t^2 = 2xyt.$$

This is quadratic in y, so it has another solution $\overline{y}$, which satisfies

$$y + \overline{y} = 2xt, \; y\overline{y} = x^2 - t^2.$$

So the transformation

$$(T1-) \qquad \underline{x} = \overline{y} = 2xt - y, \; \underline{y} = x, \; \underline{t} = t$$

gives a new solution; but it may violate the condition $0 \le x < y$. This happens if $t \ge x$.

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Continued

If $t > x$, then one of the following transformation works (similarly easy details omitted):

$$(T2-) \qquad \underline{x} = -\overline{y} = y - 2xt,\ \underline{y} = x,\ \underline{t} = t - 2x(y - 2xt),$$

$$(T3-) \qquad \underline{x} = x,\ \underline{y} = -\overline{y} = y - 2xt,\ \underline{t} = t - 2x(y - 2xt).$$

Finally if $t = x$, then we arrived at the stem $(n, 2n^2, n)$ from where both (T1–) és (T2–) go to the root.

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Concluded

Expressing the variables x, y, t by $\underline{x}, \underline{y}, \underline{t}$ we ge the inverse transformations which climb up:

$$(T1+) \qquad x = \underline{y},\ \ y = 2xt - \underline{x} = 2\underline{y}\underline{t} - \underline{x},\ \ t = \underline{t},$$

$$(T2+) \qquad x = \underline{y},\ \ y = 2\underline{y}(2\underline{x}\underline{y} - \underline{t}) - \underline{x},\ \ t = 2\underline{x}\underline{y} - \underline{t},$$

$$(T3+) \qquad x = \underline{x},\ \ y = 2\underline{x}(2\underline{x}\underline{y} - \underline{t}) - \underline{y},\ \ t = 2\underline{x}\underline{y} - \underline{t}.$$

Finally by some change of notation and the substitution $z = xy + t$ yields the transformations (T1)–(T3) of the theorem.

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Unrestricted version

$$x^2 + y^2 - t^2 = 2xyt$$

Don't asume positivity and size ordering.
Trivial trasfomations: P, exchange of x and y; change the sign of two of x, y, t, S_x, S_y, S_t, where the subscript shows the one unchanged. This generates a group of order 8, and from each group of 8 solutions exacly one satisfies the (now discrded) original restrictions.
The equation is quadratic in each variable, so it has another root in each, which satisfy

$$x + \overline{x} = 2yt,\ \ x\overline{x} = y^2 - t^2,$$
$$y + \overline{y} = 2xt,\ \ y\overline{y} = x^2 - t^2,$$
$$t + \overline{t} = -2xy,\ \ t\overline{t} = -(x^2 + y^2).$$

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The proper transformations

arise by using the other root:

$$R_x: \qquad x' = \overline{x} = 2yt - x,\ y' = y,\ t' = t,$$

$$R_y: \qquad y' = \overline{y} = 2xt - y,\ x' = x,\ t' = t,$$

$$R_t: \qquad t' = \overline{t} = -2xy - t,\ x' = x,\ y' = y.$$

Each is of order 2. S_x, S_y, S_t commute with each other and with R_x, R_y, R_t, while P permutes them: $PS_x = S_yP$, $PR_x = R_yP$.

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The sturcture of transformations

Theorem

R_x, R_y, R_t generate a group, which is the free product of three 2-element groups.

In everyday words, by applying them in any order where two consecutive ones never coincide we obtain different transformations.
Attention: we don't claim that applying them on any solution we get different triples, this would be false, just that *there is* some solution when they are different.
So we upgraded the semigroup to a group, but paid a price.

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Reason

Theorem

Let (x, y, t) be a solution with $xy \neq 0$. Then exactly one of R_x, R_y, R_t, namely which acts on the one with maximal absolute value, decreases it, the other two increase it. (There is always a single maximal one.)

By always applying the decreasing transformation we arrive at one with $xy = 0$; these are $(0, n, n)$ and variants, the *root*. From a root one incerasing transformations go to $(-2n^2, n, n)$, the *stem*. From this we always get two branches.
The solutions are the same, but the tree is different. The reason is that in the positive version we skipped the case $t < 0$, essentially by R_t.

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Parametic desciption?

Would be nice, but probably does not exist.
At least with polynomials it does not exist.

Theorem

Thee does not exist a finite collection of triples of polynomial (f_i, g_i, h_i) (in any number of variables) such that

$$x = f_i(n),\ y = g_i(n),\ t = h_i(n),\ n \in \mathbb{Z}^k$$

gives all solutions of our equation.

Reason

Use a slightly different classification of solutions.
Call those with $xy = 0$ or $t = \pm 1$ (so one is minimal possible) the *margin*. From any solution a repeated application of decreasing transformations hits the margin; call the number of steps the *level*. Level 0 is the margin, the union of the root and the main sequence.
We claim that if f, g, h are integer-valued polynomials such that

$$f^2 + g^2 - h^2 = 2fgh,$$

then all triples $(f(n), g(n), h(n))$ are in a finite number of levels.
Given a triplet (f, g, h) of polynomials, we try to apply one of the transfomations R_x, R_y, R_t so that the degree decreases. After some steps this process stops: either the degree cannot decrease, or one is the 0 polynomial.

(continued)

If $f = 0$, then $g = h$ or $g = -h$, we are in the root; similarly if $g = 0$. $h = 0$ is impossible.
If no transformation decreases the degree, then (calculations omitted) one must be constant.
Asume $f = c$. Then

$$g^2 - 2cgh - h^2 = -c^2,$$

which can be rewritten as

$$\left(g - (c + \sqrt{c^2 + 1})h\right)\left(g - (c - \sqrt{c^2 + 1})h\right) = -c^2.$$

Hence both $g - (c + \sqrt{c^2 + 1})h$ and $g - (c - \sqrt{c^2 + 1})h$ are constants, and so are g, h.
Similar calculations work when g or h is constant.
In each case the values of (f, g, h) are on a single level; we came there by a finite number of steps, so theoriginal triples are also on a finite number of levels.

Number of solutions

Let $F(N)$ be the number of solutions with $0 < x < y \leq N$.

Theorem

There are numbers $c_3, c_4, \ldots$ *such that for all* k *we have*

$$F(N) = N + \sqrt{N/2} + c_3 N^{1/2} + \ldots + c_k N^{1/k} + O\big(N^{1/(k+1)}\big).$$

Here N is the main sequence, $\sqrt{N/2}$ is the stem.

Problem: pythagorean solutions?

$x = 3, y = 4, z = 12$ has the property that $x^2 + y^2$ is also a square. What else?
There are infinitely many examples in the main sequence. This is the (almost Pell) equation

$$n^2 + (n+1)^2 = m^2.$$

All solutions can be obtained from the trivial solution $n = 0, m = 1$ by the recursion.

$$n' = 3n + 2m + 1, \;\; m' = 3m + 4n + 2$$

First the above $x = 3, y = 4$, next $x = 20, y = 21$.
Is there a solution outside the main sequence?

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The end.