The r-Fubini–Lah numbers and polynomials

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Bell numbers and polynomials

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  (G. Nyul, G. Rácz)
- $F_{n,r}$, $F_{n,r}(x)$: r-Fubini numbers and polynomials
  (I. Mező, G. Nyul)

Definition of the r-Fubini–Lah numbers

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$$FL_{n,r}(x) = \sum_{k=0}^{n} (k+r)! \left\lfloor \begin{matrix} n \\ k \end{matrix} \right\rfloor_r x^k$$

r-Fubini–Lah numbers and polynomials

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Combinatorial interpretation

$FL_{n,r}(c)$ counts the number of ordered partitions of a set with $n+r$ elements into ordered subsets and colourings of the ordered subsets with c colours such that r distinguished elements belong to distinct ordered blocks and their ordered blocks are not coloured ($c \geq 1$).

Small values of r

- If $n \geq 1$, then

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$$xFL_{n,1}(x) = FL_{n+1,0}(x),$$

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Recurrence

If $n \geq 0$ and $r \geq 1$, then

$$FL_{n,r}(x) = r\sum_{k=0}^{n}\binom{n}{k}(n-k+1)!FL_{k,r-1}(x) + x\sum_{k=0}^{n-1}\binom{n}{k}(n-k)!FL_{k,r}(x),$$

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Another recurrence for the polynomials

If $n, r \geq 0$, then

$$FL_{n+1,r}(x) = ((r+1)x + n + 2r)\, FL_{n,r}(x) + (x^2 + x)FL'_{n,r}(x).$$

Dobiński type formula

If $n, r \geq 0$, then

$$FL_{n,r}(x) = \frac{1}{(x+1)x^r} \sum_{k=0}^{\infty} (k+r)^{\overline{n}} k^{\underline{r}} \left(\frac{x}{x+1} \right)^k,$$

$$FL_{n,r} = \sum_{k=0}^{\infty} \frac{(k+r)^{\overline{n}} k^{\underline{r}}}{2^{k+1}}.$$

Exponential generating function

If $r \geq 0$, then

$$\sum_{n=0}^{\infty} \frac{FL_{n,r}(x)}{n!} y^n = \frac{r!}{(1-y)^{r-1}(1-y-xy)^{r+1}},$$

$$\sum_{n=0}^{\infty} \frac{FL_{n,r}}{n!} y^n = \frac{r!}{(1-y)^{r-1}(1-2y)^{r+1}}.$$

Connection with the *r*-Fubini numbers and polynomials

If $n, r \geq 0$, then

$$FL_{n,r}(x) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_r F_{k,r}(x),$$

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