Power sums and diophantine equations

Ákos Pintér

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This talk is dedicated to the memory of my friend, Béla Brindza (1958-2003)

Preliminaries

$$S_k(x) = 1^k + 2^k + \ldots + (x-1)^k$$

$$S_k(x) = \frac{1}{k+1}\left(B_{k+1}(x) - B_{k+1}\right),$$

where $B_k(x)$ denotes the kth Bernoulli polynomial and $B_k = B_k(0)$.

$$S_k(x) = x(x-1)(2x-1)T_k(x), \quad k \text{ is even}$$

$$S_k(x) = x^2(x-1)^2 T_k(x), \quad k > 1 \text{ is odd}$$

Earlier times

A classical example (cannonball problem):

$$1^2 + 2^2 + \ldots + (x-1)^2 = y^2$$

All the solutions are $(x, y) = (2, 1), (25, 70)$

Lucas, 1875, conjecture

Watson, 1918, proof

Earlier times, the general case, ineffective

$$S_k(x) = 1^k + 2^k + \ldots + (x-1)^k = y^n$$

k is a fixed positive integer, x, y, n are unknown integers with $x > 2, y > 1, n > 1$

Schäffer, 1956

Apart from the cases

$$(k, n) \in \{(1,2), (3,2), (3,4), (5,2)\}$$

the equation $S_k(x) = y^n$ has only finitely many solutions in x and y for every fixed exponent $n \geq 2$ and fixed $k \geq 1$.

Exceptional cases

$$S_1(x) = \frac{1}{2}x(x-1) = y^2,$$

$$S_3(x) = \frac{1}{4}x^2(x-1)^2 = y^2,$$

$$S_3(x) = \frac{1}{4}x^2(x-1)^2 = y^4,$$

$$S_5(x) = \frac{1}{12}x^2(x-1)^2(2x^2-2x-1) = y^2.$$

Generalizations, effective results

Győry, Tijdeman, Voorhoeve, 1980

Let $k \geq 2$ and r be a fixed integers with $k \notin \{3, 5\}$ if $r = 0$, and let s be a squarefree odd integer. Then the equation

$$sS_k(x) + r = y^n$$

in positive integers $x, y \geq 2, n \geq 2$ has only finitely many solutions, and all these can effectively determined.

A surprising generalization

Voorhoeve, Győry, Tijdeman, 1979

Let $R(x)$ be a fixed polynomial with integer coefficients, and let $k \geq$ be a fixed integer such that $k \notin \{3, 5\}$. Then the equation

$$S_k(x) + R(x) = y^n$$

in integers $x, y \geq 2, n \geq 2$ has only finitely many solutions, and an effective upper bound can be given for n.

Common effective generalization

Set $A = \mathbb{Z}[X], \kappa = (k+1)\prod_{p-1|(k+1)!} p$ (p prime) and

$$F(Y) = Q_m Y^m + \ldots + Q_1 Y + Q_0 \in A[Y].$$

Consider the equation

$$F(S_k(x)) = y^n$$

in integers $x, y \geq 2, n \geq 2$.

Brindza, 1984

If $Q_i(x) \equiv 0 \pmod{\kappa^i}$ for $i = 2, \ldots, m$; $Q_1(x) \equiv \pm 1 \pmod 4$, and $k \notin \{1, 2, 3, 5\}$ then all solutions of the previous equation satisfy $\max(x, y, n) < c_1$, where c_1 is an effectively computable constant depending only on F and k.

Remarks.

1. Effective by Brindza's effective LeVeque for superelliptic equations
2. If $Q_2(x) = \ldots Q_m(x) = 0$ and $Q_1(x) = s$, where s is an odd integer, then we have an effective version of the surprising generalization.

$sS_k(x) + r = y^n$ without condition

Rakaczki, 2012

Let $k > 1, r, s \neq 0$ be fixed integers. Then apart from the cases when (i) $k = 3$ and either$r = 0$ or $s + 64r = 0$, and (ii) $k = 5$ and either $r = 0$ or $s - 324r = 0$, the equation

$$s(1^k + 2^k + \ldots + (x-1)^k) + r = y^n$$

in integers $x > 0, |y| \geq 2$, and $n \geq 2$ has only finitely many solutions which can be effectively determined.

Other analogues

Urbanowicz, 1988

periodic function analogue

Dilcher, 1986

quadratic residue class character analogue
A cute equation with characters:

$$1^k - 3^k + 5^k - \ldots + (4x-3)^k - (4x-1)^k = \pm y^n$$

Dilcher's equation

Dilcher, 1986

For fixed $k \geq 3$ with $k \notin \{4, 5\}$ the equation

$$1^k - 3^k + 5^k - \ldots + (4x-3)^k - (4x-1)^k = \pm y^n$$

has only finitely many solutions in integers $x, y \geq 2$ and $n \geq 2$, with effective upper bounds for x, y, n.

Common roots of the proofs

$f(x) = y^n, x, y, n$ are unknown integers with $|y| > 1$ and $n \geq 2$

- Schinzel-Tijdeman Theorem for the exponent n
- Effective LeVeque's theorem by Brindza
- Zero-structure of f, e.g. three simple zero

Schäffer's conjecture

Schäffer's conjecture, 1956

For $k \geq 1$ and $n \geq 2$ with

$$(k, n) \notin \{(1, 2), (3, 2), (3, 4), (5, 2)\},$$

equation

$$1^k + 2^k + \ldots (x - 1)^k = y^n$$

has only one non-trivial solution, namely $(k, n, x, y) = (2, 2, 25, 70)$.

Remarks. Trivial solution $(x, y) = (2, 1)$. k, n, x, y are unknows.

The first candle

Bennett, Győry, P, 2004

For $1 \leq k \leq 11$ and

$$(k, n) \notin \{(1,2), (3,2), (3,4), (5,2)\},$$

equation

$$1^k + 2^k + \ldots + (x-1)^k = y^n$$

has only one non-trivial solution, namely $(k, n, x, y) = (2, 2, 25, 70)$.

Some special cases I

Jacobson, Walsh, P, 2003

For $n = 2$ and even values of k with $k \leq 58$, equation

$$1^k + 2^k + \ldots + (x-1)^k = y^n$$

has only the trivial solution except in the case $k = 2$, when there is the anomalous solution $(x, y) = (25, 70)$.

Some (very) special cases II

P, 2007

For odd values of k with $1 \leq k < 170$, the equation

$$1^k + 2^k + \ldots + (x-1)^k = y^{2n}$$

in positive integers x, y, n with $n > 2$ has only the trivial solution $(x, y) = (2, 1)$.

Bottlenecks for larger k

For $n > 2$: our approach is based on modular method. The modular method is based on the calculation of coefficients of several newforms. For larger k the level of the corresponding newforms is too high for computational purpose. For example: $B_{12} = -\frac{691}{2 \cdot 3 \cdot 5 \cdot 7 \cdot 13}$, $B_{16} = -\frac{3617}{2 \cdot 3 \cdot 5 \cdot 17}$

For $n = 2$: we have to calculate the fundamental solution to the Pell equation $x^2 - ay^2 = 1$ for LARGE a ($a > 10^{20}$).

An upper bound for n

P, 1997

All the solutions x, y, n to equation

$$1^k + 2^k + \ldots + (x-1)^k = y^n$$

with $x > 10^3(k/2)^{k+\frac{5}{2}}, y > 1$ and $n \geq 2$ satisfy

$$n < c_2 k \log 2k,$$

where c_2 is an effectively computable absolute constant.

Detour: Erdős-Moser conjecture

Erdős, Moser, 1950's

The unique solution of the equation

$$1^k + 2^k + \ldots + (x-1)^k = x^k$$

is $(k, x) = (1, 3)$.

Remark. If x is a solution then $x > 10^{10^9}$ (Gallot, Moree, Zudilin, 2011).

The number of solutions for $n = 2$

Brindza, P, 2000

For $k \geq 2$ even, the equation

$$1^k + 2^k + \ldots + (x-1)^k = y^2$$

has at most $\max\{c_3, 9^k\}$ solutions in integers x and y, where c_3 is an effectively computable absolute constant.

The number of solutions for $n > 2$

Brindza, P, 2000
Apart from the case $(k, n) = (3, 4)$, the equation

$$1^k + 2^k + \ldots + (x-1)^k = y^n$$

has at most $\max\{c_4, e^{3k}\}$ solutions in positive integers $x, y > 1$, and $n > 2$, where c_4 is an effectively computable absolute constant.

Remark. n is unknown.

Schäffer's conjecture for fixed x

Bérczes, Hajdu, Miyazaki, Pink, 2016

All solutions of the equation

$$1^k + 2^k + \ldots + (x-1)^k = y^n$$

in positive integers x, k, y, n with $x < 26$ and $n \geq 3$ are given by

$$(x, k, y, n) = (2, k, 1, n), (9, 3, 6, 4).$$

A variant of Schäffer's problem

Bartoli, Soydan, 2020

Let k, ℓ be fixed integers such that $k \geq 2, k \neq 3$, and $\ell \geq 2$. Then all solutions of the equation

$$(x+1)^k + (x+2)^k + \ldots + (\ell x)^k = y^n$$

in integers x, y, n with $x, y \geq 2, n \geq 2$ satisfy $\max\{x, y, n\} < c_5$, where c_5 is an effectively computable constant depending only on ℓ and k.

New age

There are only finite number of terms on the left-hand side. A typical example:
Zhang 2014, Bennett, Patel, Siksek, 2016

The only solutions to the equation

$$(x-1)^k + x^k + (x+1)^k = y^n, k \in \{2,3,4,5,6\}, x, y, n \in \mathbb{Z}, n \geq 2$$

are $(x, y, k, n) =$
$(1, \pm 3, 3, 2), (2, \pm 6, 3, 2)(24, \pm 204, 3, 2)(\pm 4, \pm 6, 3, 3)$, $(0, 0, 5, n)$.

Nirvana Coppola, Mar Curcó-Iranzo, Maleeha Khawaja, Vandita Patel, Özge Ülkem, 2023, arXiv:2306.05168

Simple zeros of Bernoulli polynomials

Győry, P, 202?

For every $k \geq 3$ and $b \in \mathbb{C}$ the polynomial $B_k(X) + b$ has at least three simple zeros apart from the cases

$$(k,b) = \left(3, \pm\frac{1}{12\sqrt{3}}\right), \left(4, \frac{1}{30}\right), \left(4, -\frac{7}{240}\right), \left(6, -\frac{1}{42}\right), \left(6, -\frac{1}{189}\right).$$

Simple zeros of Euler polynomials

Győry, P , 202?

Let $k \geq 3$ be an integer and $b \in \mathbb{C}$. Then the shifted Euler polynomial $E_k(x) + b$ has at least three simple zeros, apart from the cases
$(k, b) = \left(3, \pm\frac{1}{4}\right), \left(4, \frac{1}{4}\right), \left(4, -\frac{5}{16}\right), (5, 0), (6, -1)$.

Simple zeros of generalized Bernoulli polynomials

Győry, P. 202?

Let χ be the unique quadratic character with conductor $f = 4$ and $B_\chi^k(X)$ be the kth generalized Bernoulli polynomial belonging to χ. Let $k \geq 4$ be an integer and $b \in \mathbb{C}$. Then the shifted generalized Bernoulli polynomial $B_\chi^k(X) + b$ has at least three simple zeros, apart from the cases

$$(k, b) = (4, \mp 4), (5, -10), \left(5, \frac{25}{2}\right), (6, 0), (7, 224).$$

A generalization of Dilcher's result

Győry, P

Let r be a fixed rational number and $k \geq 3$ be a fixed integer. Apart from the cases

$$(k, r) \in \left\{(3, \pm 1), (4, 0), \left(4, \frac{-9}{2}\right), (5, 0), \left(6, \frac{125}{2}\right)\right\}$$

the equation

$$1^k - 3^k + 5^k - \ldots + (4x-3)^k - (4x-1)^k + r = by^n$$

has only finitely many solutions in integers $x, |y| > 1, n \geq 2$ and

$$\max(x, |y|, n) < c_6,$$

where c_6 is an effectively computable constant depending on k, r and b.

The exceptional cases

$$(k,r)=(3,\pm 1): -(2x\mp 1)(4x\pm 1)^2,$$

$$(k,r)=(4,0): -16x^2(8x^2-3), \textit{Dilcher}, 1986,$$

$$(k,r)=(4,-9/2): -\frac{1}{2}(16x^2-3)^2,$$

$$(k,r)=(5,0): -2x(16x^2-5)^2, \textit{Dilcher}, 1986$$

$$(k,r)=(6,125/2): -\frac{1}{2}(16x^2-5)^3.$$

The proof

The proof is based on a simple observation.

$$B_{\chi}^{k}(X) = -2^{k-2} k E_{k-1}\left(\frac{X+1}{2}\right),$$

$$B_{\chi}^{k} = -\frac{1}{2} k E_{k-1}.$$

Corollary (preliminary version)

Set $T(x) = 1^k - 3^k + 5^k - \ldots + (4x-3)^k - (4x-1)^k + r$ and $F(X) \in \mathbb{Q}[X], \deg F \geq 2$ and F is irreducible over $\mathbb{Q}$.

Győry, P, 202?

The equation $F(T(x)) = by^n$ in integers $x > 1, y > 1$ and $n \geq 2$ has only finitely many solutions, and

$$\max(x, y, n) < c_7,$$

where c_7 is an effectively computable constant depending only on k, F and b.

Homework

There is a Hungarian mathematical journal for secondary schools with a mathematical competition.

B. 4472. Prove that the sum of the squares of seven consecutive integers cannot be a perfect square.

B. 5290. Solve the following equation over the set of positive integers:

$$3^n + 4^n + \ldots + (n+2)^n = (n+3)^n.$$

Thank you for your attention!