Common values of linear recurrences related to ABC polynomials

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1. Introduction and notations

(Ankeny, Brauer and Chowla) ABC-polynomial family:
$P_u(X) = XQ(X)(X-u)+1$ with $Q(X) \in \mathbb{Z}[X]$ of degree q.

• Ankeny, Brauer and Chowla, 1956: the class number of the number field $\mathbb{Q}[X]/(P_u(X)\mathbb{Q}[X])$ can be large.
The Thue equation associated to ABC-polynomials

$$Y^{q+2}P_u(X/Y) = X(Y^qQ(X/Y))(X-uY)+Y^{q+2} = 1. \qquad (1)$$

• Bombieri, Schmidt, 1989: in degree linear upper bound for the number of solutions of Thue equations. (1) is nearly extremal.

• Mignotte and Tzanakis, 1991 as well as Mignotte 2000: solved (1) for $Q(X) = X - 1$.

• Pethő, 1991 as well as Mignotte, Pethő and Roth 1996 solved (1) for $Q(X) = (X - 1)(X + 1)$.

• Halter-Koch, Lettl, Pethő and Tichy, 1999:
If $Q(X) = (X - a_1) \cdots (X - a_n)$, $a_1, \ldots, a_n$ pairwise different integers, then (1) has in general only "obvious" solutions $(x, y) = (1, 0), (0, 1), (a_i, 1), i = 1, \ldots, n$. The proof is conditional, depends on the Land-Waldschmidt conjecture.

• Many other results on families of Thue equations: E. Thomas, Heuberger, Lettl, Tichy, Lemmermeyer, Voutier, Ziegler, etc.

Let $P(X) = X^k - p_{k-1}X^{k-1} - \ldots - p_0 \in \mathbb{Z}[X]$, and $A_0, \ldots, A_{k-1} \in \mathbb{Z}$. Then (A_n) is a linear recursive sequence with characteristic polynomial $P(X)$ and initial terms $A_0, \ldots, A_{k-1}$ if

$$A_{n+k} = p_{k-1}A_{n+k-1} + \ldots + p_0 A_n, \; n \geq 0.$$

Popular question: Find all n, m with $A_n = B_m$.

– Mignotte, 1978: If the characteristic polynomials of (A_n) and (B_n) have dominant roots, which are multiplicatively independent then $A_n = B_m$ has only finitely many effectively computable solutions. Many generalization.
– Laurent, 1987: $A_n = B_m$ *...can have infinitely many solutions* m, n *only in the "obvious" cases.*. Ineffective in general.
– Trend of the 21th century: complete solution: Luca, Ziegler, Bravo, Sanches, Gómez, Marques, Ddamulira, Pink, Togbé, etc

– Pethő and Tengely, 2025: Let $(F_n(u))$ be the lrs with initial terms $A, B, C \in \mathbb{Z}$ and with characteristic polynomial $X^3 - (u-1)X^2 - (u+2)X - 1$, Shanks cubic. If at least one of A, B, C is non-zero and $F_n(u) = F_m(u)$ holds for some $u, n, m \in \mathbb{Z}, n \neq m$ then $|n|, |m| < c$ with effective c.

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In this talk presented research was motivated by Halter-Koch, Lettl, Pethő and Tichy as well as by Pethő and Tengely.

Theorem 1. *Let* $P_u(X) = XQ(X)(X-u)+1$*, where* $Q(X) \in \mathbb{Z}[X]$ *is monic, separabel, and* $Q(0) = \pm 1$*. Denote* $(A_n(u))$ *the family of linear recursive sequences having characteristic polynomial* $P_u(X)$ *and initial terms* $A_0, \ldots, A_{q+1} \in \mathbb{Z}$*. Set* $L = \max\{|A_1|, \ldots, |A_{q+1}|\}$*, and assume* $L > 0$*. Then there exist effectively computable constants* u_0, c_0 *depending only on* $Q(X)$ *and* L *such that if*

$$|A_n(u)| = |A_m(u)| \tag{2}$$

holds for some integers u, n, m *with* $u > u_0$ *and* $n \neq m$ *then*

$$|n|, |m| < c_0.$$

Plan of the proof

1 Localization of the zeroes of $P_u(X)$
2 Properties of the Binet formula of $(A_n(u))$
3 Treatment of the "simple" cases
4 Lower bound for $|n|, |m|$ in term of u
5 Finish with "Bakery"

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Our dream was to prove similar theorem with $Q(0) \neq 0$, but our method is not strong enough for this .
Our proof is not yet complete, it depends on the convergence of some formal Puiseux expansions.*

*By Lemma 1 of W.M. Schmidt, *Eisenstein's theorem on power series expansions of algebraic functions*, Acta Arith. **56** (1990), 161–179 have the Puiseux expansion of all zeros of an equation $F(X,Y)=0$ positive radii of convergence. Hence the proof of the Theorem is complete. **25.10.2025**.

2. Localization of the zeroes of $P_u(X)$

Let $\beta_0 = 0, \beta_{q+1} = u$, and $\beta_1, \ldots, \beta_q$ the (pairwise distinct) zeroes of $Q(X)$. Our first lemma is due essentially to Ankeny, Brauer and Chowla.

Lemma 1. *If $u > u_0$ then*

(i) the zeroes $\alpha_j(u),\ 0 \le j \le q+1$ of $P_u(X)$ are pairwise distinct,

(ii) for a suitable arrangement of these roots

$$\lim_{u\to\infty} u(\alpha_i(u) - \beta_i) = b_i,\ i = 0,\dots,q+1$$

hold with

$$b_0 = Q(0)^{-1},\ b_i = \frac{1}{\beta_i Q'(\beta_i)} \neq 0,\ i = 1,\dots,q+1,$$

(iii) $P_u(X)$ is irreducible,

(iv) if, with the arrangement of (ii), β_i is real, then $\alpha_i(u)$ is real too. In particular the numbers $\alpha_0(u)$ and $\alpha_{q+1}(u)$ are real.

(v) any two distinct roots are multiplicatively independent.

The next lemma allows us to localize $\alpha_0(u)\alpha_{q+1}(u)$ quite precisely.

Lemma 2. *Let $X, X_1, \ldots, X_q$ be indeterminates and set $Q(X) = \prod_{j=1}^{q}(X - X_j)$. Denote $Q'(X)$ the derivative of $Q(X)$ with respect to X. Then*

$$\sum_{j=1}^{q} \frac{1}{X_j^2 Q'(X_j)} = (-1)^{q-1} \frac{\sum_{i=1}^{q} \prod_{\substack{j=1 \\ j \neq i}}^{q} X_j}{\prod_{j=1}^{q} X_j^2}. \tag{3}$$

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Lemma 3. *Let $X, X_1, \ldots, X_q$ be indeterminates and set $Q(X) = \prod_{j=1}^{q}(X - X_j)$. Denote $Q'(X)$ the derivative of $Q(X)$ with respect to X. Then*

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Corollary 1. *If $Q(0) = \pm 1$ then*

$$\frac{1}{|\alpha_0(u)|\alpha_{q+1}(u)} = 1 + \frac{(-1)^{q+1} Q'(0)}{u} + o\left(\frac{1}{u}\right)$$

for all $u > u_0$.

Corollary 2. *If* $|Q(0)| = 1$ *then there exist constants* $\kappa_1 \geq 1, \pi > 0, u_0$ *depending only on* $Q(X)$ *such that either*

$$1 + \frac{\pi}{2u^{\kappa_1}} < |\alpha_0(u)|\alpha_{q+1}(u) < 1 + \frac{2\pi}{u^{\kappa_1}}$$

or

$$1 + \frac{\pi}{2u^{\kappa_1}} < \frac{1}{|\alpha_0(u)|\alpha_{q+1}(u)} < 1 + \frac{2\pi}{u^{\kappa_1}}$$

hold for all $u > u_0$.

If $Q'(0) \neq 0$ then Corollary 1 yields the assertion.

Otherwise let

$$R(u,X) = Res_Y(P_u(X/Y), P_u(Y)) \in \mathbb{Z}[u,X].$$

One of its zeroes is $\alpha_0(u)\alpha_{q+1}(u)$ for all $u \in \mathbb{Z}$. The algebraic function $\alpha_0(u)\alpha_{q+1}(u)$ admits the Puiseux expansion

$$\alpha_0(u)\alpha_{q+1}(u) = \sum_{j=N}^{\infty} \pi_j u^{-j/r}$$

with $1 \le r \in \mathbb{Z}$. Comparing this with Corollary 1 we see that $N = 0, \pi_0 = 1$ and $\pi_j = 0$ for $j = 1, \ldots, r$. Let J be the smallest positive index with $\pi_J \neq 0$, then

$$\alpha_0(u)\alpha_{q+1}(u) = 1 + \frac{\pi_J}{u^{J/r}} + O\left(\frac{1}{u^{(J+1)/r}}\right).$$

If $\pi_J > 0$ then we obtain

$$1 + \frac{2\pi_J}{u^{J/r}} < |\alpha_0(u)|\alpha_{q+1}(u) < 1 + \frac{\pi_J}{2u^{J/r}}$$

for all $u \ge u_0$.

Convergence of Puiseux expansion?

3. Properties of the Binet formula of $(A_n(u))$

Lemma 4. *The sequence $(A_n(u))_{n\geq 0}$ is linear recursive for $n < 0$ too with the characteristic polynomial $X^{q+2}P_u(1/X)$.*

There exist uniquely determined functions $a_0(u), \ldots, a_{q+1}(u) \in \mathbb{Q}(\alpha_0(u), \ldots, \alpha_{q+1}(u))$, such that

$$A_n(u) = a_0(u)\alpha_0(u)^n + \ldots + a_{q+1}(u)\alpha_{q+1}(u)^n \qquad (5)$$

holds for all $n \in \mathbb{Z}$. Binet formula.

Denote $(\mathbf{b_1}, \ldots, \mathbf{b_d})$ the matrix having columns $\mathbf{b_1}, \ldots, \mathbf{b_d}$. Further set

$$
\begin{aligned}
\mathbf{A} &= (A_0, \ldots, A_{q+1})^t, \\
\mathbf{v}_j &= (\alpha_j(u)^0, \ldots, \alpha_j(u)^{q+1})^t, \ j = 0, \ldots, q+1, \\
D(u) &= \det(\mathbf{v}_0, \ldots, \mathbf{v}_{q+1}) \quad \text{and} \\
D_j(u) &= \det(\mathbf{v}_0, \ldots, \mathbf{v}_{j-1}, \mathbf{A}, \mathbf{v}_{j+1}, \ldots, \mathbf{v}_{q+1}), \ j = 0, \ldots, q+1.
\end{aligned}
$$

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D_j(u) &= \det(\mathbf{v}_0, \ldots, \mathbf{v}_{j-1}, \mathbf{A}, \mathbf{v}_{j+1}, \ldots, \mathbf{v}_{q+1}), \ j = 0, \ldots, q+1.
\end{aligned}$$

Lemma 5. *The following assertions are true for all $u \in \mathbb{Z}$*
(i) $a_j(u) \neq 0$ for all $j = 0, \ldots, q+1$,
(ii) $D(u), D_0(u), D_{q+1}(u), a_0(u)$ and $a_{q+1}(u)$ are real functions,
(iii)

$$\frac{a_{q+1}(u)}{a_0(u)} = \frac{D_{q+1}(u)}{D_0(u)}.$$

Lemma 6. *There exist easily computable constants* c_1, c_2, c_3, c_4 *depending only on* L *and on the degree and the roots of* $Q(X)$ *such that*

$$\begin{aligned} c_1\alpha_{q+1}(u)^{q+1} &\leq |D(u)| \leq c_2\alpha_{q+1}(u)^{q+1} \\ c_3\alpha_{q+1}(u)^{1-q^2} &\leq |D_j(u)| \leq c_4\alpha_{q+1}(u)^{q+1}, \quad j = 0, \ldots, q \\ c_5\alpha_{q+1}(u)^{-q(q+1)} &\leq |D_{q+1}(u)| \leq c_6 \end{aligned}$$

hold for all $u \geq u_0$.

Let

$$\mathcal{D}'_0 = \lim_{u\to\infty} D_0(u)/\alpha_{q+1}(u)^{q+1}, \quad \mathcal{D}'_{q+1} = \lim_{u\to\infty} D_{q+1}(u).$$

Because of $\beta_0(u) \to 0$ we have

$$\mathcal{D}'_{q+1} = \begin{vmatrix} 1 & 1 & \dots & 1 & A_0 \\ 0 & \beta_1 & \dots & \beta_q & A_1 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & \beta_1^{q+1} & \dots & \beta_q^{q+1} & A_{q+1} \end{vmatrix} = \beta_1 \cdots \beta_q \begin{vmatrix} 1 & \dots & 1 & A_1 \\ \beta_1 & \dots & \beta_q & A_2 \\ \dots & \dots & \dots & \dots \\ \beta_1^q & \dots & \beta_q^q & A_{q+1} \end{vmatrix}.$$

Setting

$$d_{q+1} = \begin{vmatrix} 1 & \dots & 1 & A_1 \\ \beta_1 & \dots & \beta_q & A_2 \\ \dots & \dots & \dots & \dots \\ \beta_1^q & \dots & \beta_q^q & A_{q+1} \end{vmatrix}$$

we get $\mathcal{D}'_{q+1} = (-1)^q Q(0) d_{q+1}$.

Similarly Lemma 1 (ii) yields

$$\mathcal{D}_0' = \begin{vmatrix} A_0 & 1 & \dots & 1 & 0 \\ A_1 & \beta_1 & \dots & \beta_q & 0 \\ \dots & \dots & \dots & \dots & \dots \\ A_q & \beta_1^q & \dots & \beta_q^q & 0 \\ A_{q+1} & \beta_1^{q+1} & \dots & \beta_q^{q+1} & 1 \end{vmatrix} = \begin{vmatrix} A_0 & 1 & \dots & 1 \\ A_1 & \beta_1 & \dots & \beta_q \\ \dots & \dots & \dots & \dots \\ A_q & \beta_1^q & \dots & \beta_q^q \end{vmatrix} = d_0.$$

Unfortunately $d_0 = 0$ or/and $d_{q+1} = 0$ can happen!

If $d_0, d_{q+1} \neq 0$ then the next (key) lemma can be proved with $\kappa = q + 1$ by elementary algebra and analysis.

Lemma 7. *There exist effective constants* $\kappa, \kappa_2 \in \mathbb{Q}, \varphi, \mu \in \mathbb{R}, |\varphi|, \mu > 0$ *depending only on* Q, L *such that*

$$\lim_{u \to \infty} \Phi(u) = \lim_{u \to \infty} u^{\kappa} \frac{D_{q+1}(u)}{D_0(u)} = \varphi.$$

Moreover, if $\Phi(u)$ *is not ultimately constant, then*

$$\frac{\mu}{2u^{\kappa_2}} < |\Phi(u) - \varphi| < \frac{2\mu}{u^{\kappa_2}}$$

holds for all $u \geq u_0, u \in \mathbb{Z}$.

Outline of the proof if d_0 or d_{q+1} is zero.

With $D_0(u), D_{q+1}(u)$ is $D_0(u)/D_{q+1}(u)$ too a real algebraic function. It admits a formal Puiseux expansion, i.e. there exist integers J, r and non-zero complex numbers π_J such that

$$\frac{D_{q+1}(u)}{D_0(u)} = \sum_{j=J}^{\infty} \pi_j u^{-j/r}.$$

The choice $\kappa = J/r, \varphi = \pi_J$ proves the first statement, because

$$\Phi(u) = u^{\kappa} \frac{D_{q+1}(u)}{D_u(u)} = \varphi + \sum_{j=1}^{\infty} \pi_{J+j} u^{-j/r}.$$

If $\Phi(u)$ is not ultimately constant, then there exists $j > 0$ with $\pi_{J+j} \neq 0$ too. Choosing J_0 the smallest such index, the assertion holds with $\mu = |\pi_{J+J_0}|, \kappa_2 = J_0/r$.

Lemma 8. *Assume that* $u, n \in \mathbb{Z}, n \geq 0, u > u_0$*, Set* $F = 2\max\{|\beta_1|^{\pm 1}, \ldots, |\beta_q|^{\pm 1}\}$*. If* $n \geq n_2$ *then*

$$||A_n(u)| - |a_{q+1}(u)\alpha_{q+1}(u)^n|| < c_8 F^n. \tag{6}$$

and

$$||A_{-n}(u)| - |a_0(u)\alpha_0(u)^{-n}|| < c_8 F^n. \tag{7}$$

In particular $(|A_n(u)|), (|A_{-n}(u)|)$ *are ultimately strictly monotone increasing.*

4. Proof of Theorem 1

Let u_0, u_1 be so large that
• For all $u \geq u_0$ $\alpha_{q+1}(u)$ and $1/\alpha_0(u)$ is dominant among the zeroes of $P_u(X)$ as well as of the zeroes of $X^{q+2}(1/X)$ respectively.
• For all $u \geq u_1 \geq u_0$ $\alpha_{q+1}(u)$ and $\alpha_0(u)$ are multiplicatively independent.

Our equation to be solved

$$|A_n(u)| = |A_m(u)|$$

compress three equations. Assume that $u, n, m \in \mathbb{Z}, n \neq m$ is a solution of it. Then either $n, m \geq 0$ or $n, m < 0$ or $n \geq 0, m < 0$.

4.1 Treatment of the "simple" cases

Case $n, m \geq 0$ If $u \geq u_0$ then $\alpha_{q+1}(u)$ is dominant $\longrightarrow$ $(|A_n(u)|$ is strictly monotone increasing whenever $n > n_0$ $\longrightarrow$ no solution if $n > n_1$ or $m > n_1$.

Case $n, m < 0$ The same argumentation with $\alpha_0(u)$ instead of $\alpha_{q+1}(u)$.

4.2 The "hard" case, $n \geq 0, m < 0$

Let $u_2 \geq u_1$ and $u_1 \leq u \leq u_2$ then $\alpha_{q+1}(u)$ and $1/\alpha_0(u)$ are dominant zeroes and multiplicatively independent $\longrightarrow$ (2) has finitely many effectively computable solutions by Mignotte, 1986.

If $u \geq u_0$ and $n, -m$ are large enough then (6), (7) yield

$$-c_8(F^n + F^{-m}) < \left| |a_{q+1}|\alpha_{q+1}^n - |a_0\alpha_0|^m \right| < |A_n| < c_8(F^n + F^{-m}).$$

If $|\alpha_0(u)|\alpha_{q+1}(u) > 1$ then $n < -m + c_9$, otherwise $-m < n + c_9$.

In the first case we obtain

$$\Big||a_{q+1}|\alpha_{q+1}^n - |a_0\alpha_0^m|\Big| < c_{10}F^{-m}.$$

After division by $|a_0\alpha_0^m|$ and using that F is a constant we get

$$\Big||\Phi(u)|\,(|\alpha_0|\alpha_{q+1})^{n-\kappa}|\alpha_0|^{-m-n+\kappa} - 1\Big| < |\alpha_0|^{-m/2} \qquad (8)$$

with

$$\Phi = \Phi(u) = u^\kappa \frac{a_{q+1}}{a_0} = u^\kappa \frac{D_{q+1}(u)}{D_0(u)}$$

and $\kappa \in \mathbb{Q}$.

4.3 Lower bound for $n, |m|$ in term of u

Lemma 9. *Assume that* $(n, m.u) \in \mathbb{Z}^3$ *is a solution of* (2) *such that* $n \geq 0, m < 0$. *If* $u > u_2$ *then* $n + m - \kappa \leq 0$ *and equality may only hold if* $|\Phi(u)| < 1$ *and* $\kappa_1 < \kappa_2$. *Moreover there exists an effectively computable constant* c *such that*

$$n = -m + \kappa > \begin{cases} cu^{\kappa_1 - \kappa_2}, & \text{if} \qquad\qquad \kappa_1 < \kappa_2 \\ cu \log u, & \text{otherwise.} \end{cases}$$

If $n + m - \kappa > 0$ *then* $-m > cu \log u$.

The proof is technically involved and is divided into several cases.

Case 1, $|\Phi(u)| \geq 1$. If $n+m-\kappa \geq 0$ then

$$|\Phi(u)|\,(|\alpha_0|\alpha_{q+1})^{n-\kappa}|\alpha_0|^{-m-n+\kappa} \geq |\alpha_0|\alpha_{q+1} > 1 + \frac{1}{2u^{\kappa_1}}$$

by Corollary 2, but this contradicts (8). Hence, in this case, $n+m-\kappa < 0$.

Taking logarithm of (8), after some simplification we obtain

$$\begin{aligned}(n-\kappa)\log(|\alpha_0|\alpha_{q+1}) &> (m+n-\kappa)|\log|\alpha_0|| - \log|\Phi(u)| - 2|\alpha_0|^{-m/2} \\ &> \left|\log\frac{|\alpha_0|}{e(|\varphi|+1)}\right|\end{aligned}$$

Here we used that $2|\alpha_0|^{-m/2} < 1$ and $|\Phi(u)| < |\varphi| + 1$.

Corollary 2 yields $|\alpha_0|\alpha_{q+1} < 1 + \frac{2B'}{u} < \frac{3}{2}$ as well, but then

$$\log(|\alpha_0|\alpha_{q+1}) < \frac{4B'}{u}.$$

Hence

$$-m > n - \kappa > \frac{u}{4B|}\left|\log \frac{|\alpha_0|}{e(\varphi+1)}\right| > \frac{u}{4|B'|}\log \frac{u}{2e(|\varphi|+1)}.$$

Case 2, $|\Phi(u)| < 1$.

Subcase 2a, $\kappa_1 \leq \kappa_2$ **and** $|\Phi(u)| \geq 1 - \frac{\tau}{u^{\kappa_1}}$. There is a constant d, such that $|\Phi(u)|\,(|\alpha_0|\alpha_{q+1})^d > 1 \longrightarrow$ replace Φ by this product.

Subcase 2b, $\kappa_1 > \kappa_2$. If $n + m - \kappa = 0$ and $n < cu^{\kappa_1-\kappa_2}$ then $\left||\Phi(u)|\,(|\alpha_0|\alpha_{q+1})^{n-\kappa} - 1\right|$ contradicts (8).

4.4 Final argumentation

We know

$$|\Gamma(u) - 1| < |\alpha_0(u)|^{-m/2} \tag{9}$$

with

$$\Gamma(u) = \left|\frac{a_{q+1}(u)}{a_0(u)}\right| |\alpha_0(u)|^m \alpha_{q+1}(u)^n,$$

$-m > n + c_9$ and $-m > cu^f$ with $c, c_9, f = \kappa_1 - \kappa_2 > 0$ constants.

If $u > u_0$ then $\Gamma(u) \neq 1$: simple $\longrightarrow$ apply Bakery for fixed $u > u_2$.

Lemma 10 (Matveev,2000)**.** *Let* $\mathbb{K}$ *be an algebraic number field of degree* $d_{\mathbb{K}}$ *and let* $\eta_1, \eta_2, \ldots, \eta_t \in \mathbb{K} \setminus \{0\}$*, and* $e_1, \ldots, e_t$ *be nonzero integers. Put*

$$E = \max\{|e_1|, \ldots, |e_t|, 3\} \qquad \text{and} \qquad \Gamma = \prod_{i=1}^{t} \eta_i^{e_i}.$$

Let $F_1, \ldots, F_t$ *be such that*

$$F_j \geq \max\{d_{\mathbb{K}} h(\eta_j), |\log \eta_j|, 0.16\}, \quad \text{for} \quad j = 1, \ldots t.$$

If $\Gamma \neq 1$*, then*

$$\log |\Gamma - 1| > -3 \cdot 30^{t+4} (t+1)^{5.5} d_{\mathbb{K}}^2 (1 + \log d_{\mathbb{K}})(1 + \log tE) F_1 F_2 \cdots F_t.$$

In our case $t = 3$, $\mathbb{K} = \mathbb{Q}(\alpha_0(u), \ldots, \alpha_{q+1}(u))$, hence $d_{\mathbb{K}} \leq (q+2)!$,

$$\eta_1 = \left|\frac{a_{q+1}(u)}{a_0(u)}\right|,\ \eta_2 = |\alpha_0(u)|,\ \eta_3 = \alpha_{q+1}(u),$$

and $e_1 = 1, e_2 = m, e_3 = n$ and $E = \max\{n, |m|\} = |m| - c_9$.

Plainly $h(\eta_2) = h(\eta_3)$, and

$$\begin{aligned} h(\eta_3) &= h(\alpha_{q+1}) = \frac{1}{q+2}\left(\log(\alpha_{q+1}) - \log(|\alpha_0|) + \sum_{j=1}^{q} \max\{1, \log|\alpha_j|\}\right) \\ &\leq c_{14} \log(\alpha_{q+1}). \end{aligned}$$

$$\begin{aligned} h(\eta_1) &= h\left(\frac{a_{q+1}(u)}{a_0(u)}\right) = h\left(\frac{D_{q+1}(u)}{D_0(u)}\right) \\ &\leq h(D_{q+1}(u)) + h(D_0(u)) \leq c_{15} \log^{q+1}(\alpha_{q+1}(u)). \end{aligned}$$

The choice $F_1 = F_2 = F_3 = c_{16} \log^{q+1}(\alpha_{q+1}(u))$ is allowed, and Matveev's theorem yields

$$\begin{aligned} \log(|\Gamma| - 1) &> -c_{17}(1 + \log(3|m|))c_{16}^3 \log^{3(q+1)}(\alpha_{q+1}(u)) \\ &> -c_{18}(\log|m|) \log^{3(q+1)}(\alpha_{q+1}(u)). \end{aligned}$$

In c_{17} we incorporated that $t = 3$ and the estimations for $d_{\mathbb{K}}$, which depends only on q.

Comparing this with the upper bound (9) we obtain

$$\frac{|m|}{\log|m|} < 2c_{18} \log^{3q+2}(\alpha_{q+1}(u)),$$

which implies

$$|m| < c_{19} \log^{3(q+1)}(\alpha_{q+1}(u))$$

after some simple and obvious transformation.

On the other hand Lemma 9 gives us a lower bound for $|m|$, which yields

$$cu^f < |m| < c_{19} \log^{3(q+1)}(\alpha_{q+1}(u)),$$

with positive constants c, f. This inequality yields $u < c_{20}$, but the $|m|$ and n are bounded too.

Remarks and problems

• If $d_0 \neq 0$ and $d_{q+1} \neq 0$ then the proof is complete, otherwise it depends on the convergence assumption of the formal Puiseux expansion.

• Does there exist $Q(X) \in \mathbb{Z}[X]$ separabel, $Q(0) = \pm 1$, and with leading coefficient 1 such that $P_u(X)$ or $X^{q+2}P_u(1/X)$ does not have dominant root? Yes, then there is presently no effective estimate.

• Does there exist $Q(X) \in \mathbb{Z}[X]$ separabel, $Q(0) = \pm 1$, and with leading coefficient 1 such that $P_u(X)$ has multiplicatively dependent roots? Yes, then our equation may have infinitely many solutions!

• What if $|Q(0)| > 1$?

Thank you
for your attention!