A kit for linear forms in three logarithms

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History

- Hilbert's 7th problem (1900):
  is α^β transcendental for algebraic $\alpha \neq 0, 1$ and algebraic $\beta \notin \mathbb{Q}$?
  Yes: Gelfond and Schneider, independently (1934).
- Equivalent to
  $$\beta \log(\alpha) + \log(\gamma) \neq 0,$$
  for all algebraic numbers, γ.
  This is a linear form in the logarithms of (two) algebraic numbers.
- if $\beta \log(\alpha) + \log(\gamma) \neq 0$, can we find lower bound for
  $$|\beta \log(\alpha) + \log(\gamma)|?$$
- Yes: Gelfond (1949).
- Question (Gelfond): how about a lower bound for
  $$|b_1 \log(\alpha_1) + \cdots + b_n \log(\alpha_n) + \log(\gamma)|?$$
- Yes: Baker (1966).

Applications

- find all imaginary quadratic fields with class number 1,
- effective irrationality measures for arbitrary real algebraic numbers,
- bounds for size of integer solutions of Thue equations,
- same for elliptic, hyper- and super-elliptic equations,

  ...
- Key for last two: can be turned into unit equations in two variables: $au_1 + bu_2 = c$ where u_1, u_2 are variables, units in a number field.
- See Y. Bugeaud, "Linear Forms in Logarithms and Applications".

linear forms in two logs

- one can also use Schneider's approach too.
- Applied initially to linear forms in two logs: $b_1 \log \alpha_1 + b_2 \log \alpha_2$
  Mignotte and Waldschmidt (1978–1989)
- Laurent introduced his interpolation determinants.
  Improved estimates (1995 with Mignotte and Nesterenko; 2008).
- Lots of applications.
  1995 paper has 139 citations. 2008 paper has 70.

linear forms in three logs

- Gelfond-Linnik (1948): effectively computable bounds for imaginary quadratic fields with class number 1.
- Tijdeman (1976): effectively computable bounds for Catalan's conjecture: $x^p - y^q = 1$.
- Pethö and Shorey & Stewart (1982/3): effective proof that only finitely many perfect powers in any binary recurrence sequence.
- Bugeaud, Mignotte and Siksek (2006): 0, 1, 8 and 144 are the only perfect powers in the Fibonacci sequence.

Proof Outline

- interpolation determinant, Δ
  produces a multi-variable polynomial
- zero estimate gives conditions for polynomial to be non-zero
- Liouville gives lower bound for $|\Delta|$.
- assume linear form is small, analysis gives upper bound for $|\Delta|$.
- Upper and lower bounds for $|\Delta|$ contradict each other:
  lower bound for linear form.

Kit Outline

Context: have a problem that reduces to a linear form in 3 logs.

(1) obtain an upper bound for linear form in three logs.

(2) combine upper bound in (1) with lower bound of Matveev, obtain upper bound, B_1, for quantity associated with linear form.

(3) suppose linear form in three logs is *non-degenerate*, use interpolation determinants approach and upper bound B_1 to obtain second upper bound, B_2.

If $B_2 < B_1$, we proceed to step (4).

(4) suppose linear form in three logs is *degenerate*, consider it as a linear form in two logs, apply Laurent (2008) to it, and use upper bound B_1, to get a third upper bound B_3.

New upper bound: $B_4 = \min\{B_1, \max\{B_2, B_3\}\}$.

(5) repeat steps (3) and (4) with B_4 in place of B_1: make upper bound as small as possible.

Our linear forms

- Three distinct non-zero algebraic numbers α_1, α_2 and α_3, positive rational integers b_1, b_2, b_3 with $\gcd(b_1, b_2, b_3) = 1$, and the linear form
$$\Lambda = b_1 \log \alpha_1 + b_2 \log \alpha_2 - b_3 \log \alpha_3.$$
- **the real case**: α_1, α_2 and α_3 are real numbers greater than 1, and the logarithms of the α_i's are all real and positive.
- **the imaginary case**: α_1, α_2 and α_3 are complex numbers $\neq 1$ of modulus one.
  Note the logarithms will be purely imaginary.
- Not a restriction since
$$|\Lambda| \geq \max\{|\mathrm{Re}(\Lambda)|, |\mathrm{Im}(\Lambda)|\}.$$

Our matrices

- Let
$$\Lambda = b_1 \log(\alpha_1) + b_2 \log(\alpha_2) - b_3 \log(\alpha_3).$$
- K, L, R, S, T are positive rational integers with $K, L \geq 2$.
$N = K(K+1)L/2$ and we assume that $RST \geq N$.
- $d_1 = \gcd(b_1, b_3)$ and $d_2 = \gcd(b_2, b_3)$, put
$b_1 = d_1 b_1'$, $b_2 = d_2 b_2''$, $b_3 = d_1 b_3' = d_2 b_3''$.
$$\Delta = \det\left(\binom{r_j b_3' + t_j b_1'}{k_i} \binom{s_j b_3'' + t_j b_2''}{m_i} \alpha_1^{\ell_i r_j} \alpha_2^{\ell_i s_j} \alpha_3^{\ell_i t_j} \right),$$
where (k_i, m_i, ℓ_i) runs through all triples of integers with
$0 \leq k_i, m_i$, $0 \leq k_i + m_i \leq K - 1$ and $0 \leq \ell_i \leq L - 1$.
r_j, s_j and t_j are non-negative integers less than R, S and T, respectively, such that (r_j, s_j, t_j) runs over N distinct triples.
- If $\Delta = 0$, then $P\left(rb_3' + tb_1', sb_3'' + tb_2'', \alpha_1^r \alpha_2^s \alpha_3^t\right) = 0$ for all (r, s, t).

Zero estimate

Proposition (Gouillon (2003))

Let $\mathbb{K}$ *be an algebraically closed field with* $\mathrm{char}(\mathbb{K}) = 0$. *Suppose that* K *and* L *are positive integers and that* Σ_1, Σ_2 *and* Σ_3 *are non-empty finite subsets of* $\mathbb{K}^2 \times \mathbb{K}^\times$ *such that*

$$\begin{cases} \mathrm{Card}\,\{\lambda x_1 + \mu x_2 : (x_1, x_2, y) \in \Sigma_1\} & > K, \\ \mathrm{Card}\,\{y : (x_1, x_2, y) \in \Sigma_1\} & > L, \end{cases}$$

$$\begin{cases} \mathrm{Card}\,\{(\lambda x_1 + \mu x_2, y) : (x_1, x_2, y) \in \Sigma_2\} & > 2KL, \\ \mathrm{Card}\,\{(x_1, x_2) : (x_1, x_2, y) \in \Sigma_2\} & > K^2, \end{cases}$$

for all $(\lambda, \mu) \in \mathbb{K}^2 \setminus \{(0,0)\}$, *and also that*

$$\mathrm{Card}\,\Sigma_3 > 3K^2 L.$$

If $0 \neq P \in \mathbb{K}[X_1, X_2, Y]$ *with* $\deg_{X_1}(P) + \deg_{X_2}(P) \leq K$ *and* $\deg_Y(P) \leq L$, *then* P *does not vanish on all of* $\Sigma_1 + \Sigma_2 + \Sigma_3$.

Zero estimate application

- Define our Σ_j's:

  $$\Sigma_j = \left\{ \left(r + t\beta_1, s + t\beta_2, \alpha_1^r \alpha_2^s \alpha_3^t \right) : 0 \leq r \leq R_j, 0 \leq s \leq S_j, 0 \leq t \leq T_j \right\},$$

  where $R_j, S_j, T_j \in \mathbb{Z}_{>0}$, $\beta_1 = b_1/b_3$ and $\beta_2 = b_2/b_3$.
- Use Gouillon's zero estimate to get conditions on our parameters.
- If, for some positive real number χ,
  - (i) $(R_1 + 1)(S_1 + 1)(T_1 + 1) > K \cdot \max\{R_1 + S_1 + 1, S_1 + T_1 + 1, R_1 + T_1 + 1, \chi\mathcal{V}\}$,
  - (ii) Card $\{\alpha_1^r \alpha_2^s \alpha_3^t : 0 \leq r \leq R_1,\ 0 \leq s \leq S_1,\ 0 \leq t \leq T_1\} > L$,
  - (iii) $(R_2 + 1)(S_2 + 1)(T_2 + 1) > K^2$,
  - (iv) Card $\{\alpha_1^r \alpha_2^s \alpha_3^t : 0 \leq r \leq R_2, 0 \leq s \leq S_2, 0 \leq t \leq T_2\} > 2KL$ and
  - (v) $(R_3 + 1)(S_3 + 1)(T_3 + 1) > 3K^2 L$,

  all hold, then either $\Delta \neq 0$ or a degeneracy occurs.
- $R = R_1 + R_2 + R_3 + 1$, $S = S_1 + S_2 + S_3 + 1$ and $T = T_1 + T_2 + T_3 + 1$.

Liouville estimate

Let $f(X) \in \mathbb{Z}[X]$ with $\deg(f) = d$ and suppose $f(p/q) \neq 0$ for $p/q \in \mathbb{Q}$. Then $|f(p/q)| \geq 1/|q|^d = \exp(-d \log|q|)$.

Lemma

If α_1, α_2 and α_3 be non-zero algebraic numbers and $f \in \mathbb{Z}[X_1, X_2, X_3]$ such that $f(\alpha_1, \alpha_2, \alpha_3) \neq 0$, then

$$|f(\alpha_1, \alpha_2, \alpha_3)| \geq |f|^{-\mathcal{D}+1} (\alpha_1^*)^{d_1} (\alpha_2^*)^{d_2} (\alpha_3^*)^{d_3} \times \exp\{-\mathcal{D}(d_1 \operatorname{h}(\alpha_1) + d_2 \operatorname{h}(\alpha_2) + d_3 \operatorname{h}(\alpha_3))\},$$

where $\mathcal{D} = [\mathbb{Q}(\alpha_1, \alpha_2, \alpha_3) : \mathbb{Q}] \,/\, [\mathbb{R}(\alpha_1, \alpha_2, \alpha_3) : \mathbb{R}]$, $d_i = \deg_{X_i} f$, $i = 1, 2, 3$,

$$|f| = \max\{|f(z_1, z_2, z_3)| : |z_i| \leq 1, \; i = 1, 2, 3\},$$

$\operatorname{h}(\alpha)$ *is the absolute logarithmic height of* α *and* $\alpha^* = \max\{1, |\alpha|\}$.

Lower Bound for $|\Delta|$

$$g = \frac{1}{4} - \frac{N}{12RST}, \quad G_1 = \frac{NLR}{2}g, \quad G_2 = \frac{NLS}{2}g, \quad G_3 = \frac{NLT}{2}g,$$

$$M_1 = \frac{L-1}{2}\sum_{j=1}^{N} r_j, \qquad M_2 = \frac{L-1}{2}\sum_{j=1}^{N} s_j, \qquad M_3 = \frac{L-1}{2}\sum_{j=1}^{N} t_j,$$

$$b = \left(b_3'\left(\frac{R-1}{2} + \beta_1\frac{T-1}{2}\right)\right)\left(b_3''\left(\frac{S-1}{2} + \beta_2\frac{T-1}{2}\right)\right)\left(\prod_{k=1}^{K-1}(k!)^{K-k}\right)^{-\frac{12}{K(K-1)(K+1)}}.$$

Proposition

If $\Delta \neq 0$, *then*

$$\log|\Delta| \geq -\frac{\mathcal{D}-1}{3}(K-1)N\log(b) + \sum_{i=1}^{3}(M_i + G_i)\log|\alpha_i|$$

$$-2\mathcal{D}\sum_{i=1}^{3} G_i\,\mathrm{h}(\alpha_i) - \frac{\mathcal{D}-1}{2}N\log(N).$$

Upper Bound for $|\Delta|$: Overview

- We can write

$$\Delta = {\alpha_1}^{M_1}{\alpha_2}^{M_2}{\alpha_3}^{M_3}\sum_{\mathcal{I}\subseteq\mathcal{N}}(\Lambda')^{N-|\mathcal{I}|}\Delta_{\mathcal{I}},$$

  where $\mathcal{I}$ runs over all subsets of $\mathcal{N} = \{1,\ldots,N\}$, Λ' is "almost" Λ and $\Delta_{\mathcal{I}} = \Psi_{\mathcal{I}}(1)$ where $\Psi_{\mathcal{I}}(x)$ is a determinant (and an analytic function).

- Schwarz' Lemma: for $\rho > 1$,

$$|\Psi_{\mathcal{I}}(1)| \leq \rho^{-J_{\mathcal{I}}} \cdot \max_{|x|=\rho} |\Psi_{\mathcal{I}}(x)|,$$

  where $J_{\mathcal{I}} = \mathrm{ord}_{x=0}(\Psi_{\mathcal{I}}(x))$.

- Assume that $\Lambda' < \rho^{-KL}$ and obtain upper bound for $|\Delta|$.

Upper Bound for $|\Delta|$: Result

Proposition

Suppose K and L are two integers satisfying $K \geq 3$ and $L \geq 5$. If

$$\Lambda' < \rho^{-KL}$$

holds for some real number $\rho \geq 2$, then

$$\log|\Delta| < \sum_{i=1}^{3} M_i \log|\alpha_i| + \rho \sum_{i=1}^{3} G_i |\log \alpha_i| + \frac{N}{3}(K-1)\log b$$
$$- \frac{N^2}{2K}\left(1 - \frac{2}{3L} - \frac{2}{3KL} - \frac{1}{3L^3} - \frac{16}{3K^2L}\right)\log \rho$$
$$+ \log(N!) + N\log 2 + 0.001.$$

Synthesis

Proposition

With the previous notation, if $K \geq 3$, $L \geq 5$, $\rho \geq 2$, and if $\Delta \neq 0$ then

$$\Lambda' \geq \rho^{-KL}$$

provided that

$$\left(\frac{KL}{2} + \frac{L}{2} - 0.37K - 2\right) \log \rho \geq \frac{2\mathcal{D}(K-1)\log b}{3} + gL\left(a_1 R + a_2 S + a_3 T\right) + (\mathcal{D}+1)\log N,$$

where the a_i are positive real numbers which satisfy

$$a_i \geq \rho\left|\log \alpha_i\right| - \log\left|\alpha_i\right| + 2\mathcal{D}\,\mathrm{h}\left(\alpha_i\right) \qquad \text{for } i = 1, 2, 3.$$

Laurent et al:

$$K(L-1)\log \rho \geq \mathcal{D}(K-1)\log b + gL\left(a_1 R + a_2 S\right) + (\mathcal{D}+1)\log N,$$

Degeneracies

- At least one of the following conditions (**C1**) or (**C2**) holds.
  (**C1**)
  $|b_1| \leq \max\{R_1, R_2\}$, $|b_2| \leq \max\{S_1, S_2\}$ and $|b_3| \leq \max\{T_1, T_2\}$.

  (**C2**) There exist $u_1, u_2, u_3 \in \mathbb{Z}$, not all zero, such that

  $$u_1 b_1 + u_2 b_2 + u_3 b_3 = 0,$$

  with $\gcd(u_1, u_2, u_3) = 1$,

  $$|u_1| \leq \frac{(S_1+1)(T_1+1)}{\mathcal{M} - \max\{S_1, T_1\}}, \qquad |u_2| \leq \frac{(R_1+1)(T_1+1)}{\mathcal{M} - \max\{R_1, T_1\}} \quad \text{and}$$
  $$|u_3| \leq \frac{(R_1+1)(S_1+1)}{\mathcal{M} - \max\{R_1, S_1\}}.$$

  Here

  $$\mathcal{M} = \max\{R_1 + S_1 + 1, S_1 + T_1 + 1, R_1 + T_1 + 1, \chi\mathcal{V}\},$$

  where

  $$\mathcal{V} = ((R_1+1)(S_1+1)(T_1+1))^{1/2}.$$

- Use to reduce to linear form in two logs. Apply Laurent (2008).

Parameter Choice

- Four key parameters: L, m, ρ and χ.
- $K = \lfloor mLa_1a_2a_3 \rfloor$.
- $R_j = \lfloor c_j a_2 a_3 \rfloor, \quad S_j = \lfloor c_j a_1 a_3 \rfloor, \quad T_j = \lfloor c_j a_1 a_2 \rfloor$.

$$
\begin{aligned}
c_1 &= \max\left\{2^{1/3}, (\chi mL)^{2/3}, \left(\frac{2mL}{a}\right)^{1/2}\right\}, \\
c_2 &= \max\left\{(mL)^{2/3}, \sqrt{m/a}\, L\right\}, \\
c_3 &= \left(3m^2\right)^{1/3} L, \\
a &= \min\left\{a_1, a_2, a_3\right\}.
\end{aligned}
$$

- Question: how to choose L, m, ρ and χ?
- Brute force search.
- Good news: We have Pari code for this. Pari code to share!

Example

- Bennett, Győry, Mignotte, Pintér (2006): found all solutions of $AX^n - BY^n = \pm 1$ where $n \geq 3$ and A, B are S-units for $S = \{p, q\}$ with $2 \leq p < q \leq 13$.

  linear forms in logs for:
  $2^\alpha X^n - 5^\beta Y^n = 1$ where $2 \leq \alpha \leq 3$, $1 \leq \beta \leq n - 1$.

  Old kit: $n < 59 \cdot 10^6$.

  New kit: $n < 17.5 \cdot 10^6$.

iteration	bound for n	L	m	ρ	χ	new bound for n
1	$5.4 \cdot 10^{11}$	87	12.5	7.5	0.8	$41 \cdot 10^6$
2	$41 \cdot 10^6$	56	12.0	8.0	1.075	$19.1 \cdot 10^6$
3	$19.1 \cdot 10^6$	55	16.0	7.0	1.1	$17.8 \cdot 10^6$
4	$17.8 \cdot 10^6$	55	16.0	7.0	1.125	$17.5 \cdot 10^6$

Non-degenerate case alone: $n < 11.5 \cdot 10^6$.

Closing

- The Pari code: `https://github.com/PV-314/lfl3-kit`
- The preprint: `https://arxiv.org/abs/2205.08899`
- Please contact us with any questions, issues or suggestions.

We are **very** happy to help.

Thank You

Dekitifying Issues

Whack-a-mole!

- an alternative degenerate case approach due to Waldschmidt.
- good for the non-degenerate case: not good for degenerate case. And vice versa.
- Consequence: weaker results. But in progress.