Nóra Varga

On polynomials with only rational roots

Joint work with **Lajos Hajdu** and **Robert Tijdeman**

Number Theory and Algebra Seminar
Debrecen, 31 March 2023

- **Introduction**
- New theorems
- Proofs and lemmas

Introduction

Polynomials in $\mathbb{Z}[x]$ with only rational roots:

- Evertse and Győry (2015, 2017)
- Hajdu and Tijdeman (submitted)

Inroduction

If the coefficients belong to the set

- $\{-1, 1\}$: Littlewood polynomials
  - Borwein, Choi, Ferguson, Jankauskas (2015)
  - Berend and Golan (2006)
- $\{0, 1\}$: Newman polynomials (assuming that the constant term is not zero)
  - Odlyzko and Poonen (1993)
  - Mercer (2012)
- $\{-1, 0, 1\}$:
  - Borwein and Pinner (1997)
  - Borwein and Erdélyi (1997)
  - Drungilas and Dubickas (2009)

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- We generalize this result in two ways.

- Introduction
- **New theorems**
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Theorem 1

In the first generalization we require that the coefficient of f are bounded.

Theorem (Hajdu, Tijdeman, V, 202?)

Let $f(x) \in \mathbb{Z}[x]$ be a polynomial of degree n with only non-zero rational roots and height bounded by $H \geq 2$. Then we have both

$$n \leq \left(\frac{2}{\log 2} + o(1)\right) \log H \qquad (H \to \infty) \tag{2}$$

and

$$n \leq \frac{5}{\log 2} \log H. \tag{3}$$

Further, the constants $2/\log 2$ and $5/\log 2$ in (2) and (3), respectively, are best possible.

(By the height of a polynomial with integer coefficients we mean the maximum of the absolute values of its coefficients.)

Theorem 2

The second generalization concerns the case that none of the coefficients of $f(x)$ is divisible by 2 or 3.

Theorem (Hajdu, Tijdeman, V, 202?)

Every polynomial $f(x) \in \mathbb{Z}[x]$ with only rational roots of which no coefficient is divisible by 2 or 3 has degree at most 3.

Theorem 3

A further restriction is that the coefficients of f are integral S-units, that is integers composed of primes from a finite set S. Such polynomials are called S-polynomials.

The next theorem shows that for any n there are only finitely many families of S-polynomials of degree n having only rational roots.

Theorem 3

Theorem (Hajdu, Tijdeman, V, 202?)

Let S be a finite set of primes with $|S| = s$ and n a positive integer. There exists an explicitly computable constant $C = C(n, s)$ depending only on n and s and sets T_1, T_2 with $\max(|T_1|, |T_2|) \leq C$ of n-tuples of S-units and $(n-1)/2$-tuples of S-units for n odd, respectively, such that

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(i) *$(q_1, \dots, q_n) = u(r_1, \dots, r_n)$ with some $(r_1, \dots, r_n) \in T_1$ and S-unit u,*

(ii) *$n = 2t + 1$ is odd, and re-indexing $q_1, \dots, q_n$ if necessary, we have $q_1 = u$ and $(q_2, \dots, q_n) = v(r_1, -r_1, \dots, r_t, -r_t)$ with some $(r_1, \dots, r_t) \in T_2$ and S-units u, v.*

Further, the possibilities (i) *and* (ii) *cannot be excluded.*

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Remark. Observe that the rational roots of an S-polynomial $f(x)$ are S-units, i.e. rational numbers whose numerators and denominators are composed exclusively of primes in S.
This follows from the well-known fact that the denominator of a root of $f(x)$ divides the leading coefficient of $f(x)$, while its numerator divides the constant term of $f(x)$.

Proof of Theorem 1

To get (2):

On the one hand, let $f(x) = \sum_{j=0}^{n} a_j x^j$. Then

$$|f(\mathrm{i})| \leq \left| \sum_{j \text{ is even}} |a_j| + \mathrm{i} \sum_{j \text{ is odd}} |a_j| \right| \leq \sqrt{\frac{1}{2}n^2 + n + 1}\, H. \qquad (4)$$

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On the other hand, we may write $f(x) = \prod_{j=1}^{n}(q_j x - p_j)$ with $p_j, q_j \in \mathbb{Z}_{\neq 0}$ for all j. Then

$$|f(\mathrm{i})| = \left| \prod_{j=1}^{n} (q_j \mathrm{i} - p_j) \right| = \prod_{j=1}^{n} \sqrt{q_j^2 + p_j^2} \geq (\sqrt{2})^n. \tag{5}$$

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Therefore,

$$n \log 2 \leq \log\left(\frac{1}{2}n^2 + n + 1\right) + 2 \log H. \tag{6}$$

Proof of Theorem 1

The constant $2/\log 2$ in (2) is best possible:

For the height H of the polynomial $f(x) = (x^2 - 1)^{n/2}$ with even $n \geq 2$ by Stirling's formula we have $\log H = (1 + o(1))n \log 2/2$.

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To prove (3): observe that assuming $(5/\log 2)\log H < n$ from (6) we obtain

$$n \log 2 < \log\left(\frac{1}{2}n^2 + n + 1\right) + \frac{2n \log 2}{5}.$$

Hence we easily get

$$n \leq 9.$$

Further, observe that if we assume that f has a root different from ± 1, then (5) can be sharpened to

$$|f(\mathrm{i})| \geq \sqrt{5}(\sqrt{2})^{n-1}. \quad (7)$$

Thus, in this case combining $(5/\log 2)\log H < n$ with (4) and (7), we get a contradiction for $n \geq 1$.

Proof of Theorem 1

So to prove (3), we only need to check the polynomials of the shape $f(x) = \pm(x+1)^a(x-1)^{n-a}$ with $0 \leq a \leq n$ for $1 \leq n \leq 9$. A simple calculation gives that for all these polynomials (3) holds.

In particular, for $n = 5$ and $a = 2, 3$ we have equality. Thus e.g. the polynomial

$$(x-1)^3(x+1)^2 = x^5 - x^4 - 2x^3 + 2x^2 + x - 1$$

shows that **the constant** $5/\log 2$ **in** (3) **is best possible**. So the theorem is proved.

Proof of Theorem 2 - Lemma 1

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Lemma (Fine, 1947)

Let n be a positive integer such that all the coefficients of $(x+1)^n$ are odd. Then n is of the shape $2^\alpha - 1$ with some $\alpha \in \mathbb{Z}_{\geq 0}$.

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e.g.

$$(x+1)^3 = x^3 + 3x^2 + 3x + 1$$

$$(x+1)^7 = x^7 + 7x^6 + 21x^5 + 35x^4 + 35x^3 + 21x^2 + 7x + 1$$

Proof of Theorem 2 - Lemma 2

The next lemma is new, and provides a similar result for prime 3.

Lemma (Hajdu, Tijdeman, V, 202?)

Let a, b be non-negative integers. Put $n := a + b$. If none of the coefficients of $(x-1)^a(x+1)^b$ is divisible by 3, then n is of the shape $3^\beta - 1, 2 \cdot 3^\beta - 1, 3^\gamma + 3^\delta - 1$ or $2 \cdot 3^\gamma + 3^\delta - 1$ with $\beta \geq 0, \gamma > \delta \geq 0$.

Proof of Theorem 2 - Lemma 2

In the proof of Lemma 2 we call a pair of non-negative integers (a, b) *good* if none of the coefficients of

$$f_{(a,b)}(x) := (x-1)^a(x+1)^b$$

is divisible by 3; otherwise we say that (a, b) is *bad*. Observe that this property is symmetric in a and b in view of the substitution $x \to -x$.

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We have 7 cases.
e.g. CASE $a \equiv 2 \pmod 3$, $b \equiv 0 \pmod 3$. Writing $a = 3u + 2$, $b = 3v$ we see that

$$f_{(a,b)}(x) \equiv (x^3-1)^u(x^3+1)^v(x^2+x+1) \pmod 3.$$

This shows that (a, b) is good if and only if (u, v) is good.

Proof of Theorem 2 - steps

Let S be as in the statement, and let $f(x)$ be an S-polynomial with rational roots, of degree n.

Since we argue modulo 2 and 3, and $2, 3$ do not divide the leading coefficient of f, we may assume that f is monic.

Since the roots of f are odd, Lemma 1 shows that $n + 1$ is a power of 2.

Further, since the roots of f are not divisible by 3, by Lemma 2 we get that $n + 1$ is of the shape $3^\beta, 2 \cdot 3^\beta, 3^\gamma + 3^\delta$ or $2 \cdot 3^\gamma + 3^\delta$.

The combination is possible only for $n = 0, 1, 3$, as a simple check reveals.

Proof of Theorem 3 - Lemma 3

For the proof of Theorem 3 we use the theory of S-unit equations. Let S be a finite set of primes, $b_1, \dots, b_m$ non-zero rationals, and consider the equation

$$b_1 x_1 + \cdots + b_m x_m = 0 \quad \text{in } S\text{-units } x_1, \dots, x_m. \tag{8}$$

A solution $(y_1, \dots, y_m)$ of (8) is called **non-degenerate** if

$$\sum_{i \in I} b_i y_i \neq 0 \quad \text{for each non-empty subset } I \text{ of } \{1, \dots, m\}.$$

Further, two solutions $(y_1, \dots, y_m)$ and $(z_1, \dots, z_m)$ are called **proportional**, if there is an S-unit u such that $(z_1, \dots, z_m) = u(y_1, \dots, y_m)$.

Proof of Theorem 3 - Lemma 3

Lemma (Amoroso, Viada, 2009)

Equation (8) has at most $(8m-8)^{4(m-1)^4(m+s)}$ non-degenerate, non-proportional solutions, where $s = |S|$.

Remark. The original result of Amoroso and Viada concerns the inhomogeneous case, i.e. where the right hand side of (8) is 1. However, it is easy to transform their result into the shape of (8).

Proof of Theorem 3

Suppose that $f(x) = \sum_{j=0}^{n} a_j x^j$ is an S-polynomial of degree n having only rational roots $q_1, \dots, q_n$. By our assumption, $a_0, a_1, \dots, a_n$ are integral S-units. We have

$$A_j = \sigma_j(q_1, \dots, q_n) \quad (1 \leq j \leq n) \tag{9}$$

where $A_j = (-1)^j a_{n-j}/a_0$ and σ_j is the j-th elementary symmetric polynomial (of degree j) of $q_1, \dots, q_n$. Using (9) for $j = 1, 2$ we get

$$q_1^2 + \dots + q_n^2 = A_1^2 - 2A_2. \tag{10}$$

This shows that $(q_1^2, \dots, q_n^2, A_1^2, A_2)$ yields a solution to the S-unit equation

$$x_1 + \dots + x_n - x_{n+1} + 2x_{n+2} = 0. \tag{11}$$

Proof of Theorem 3

If $(q_1^2, \ldots, q_n^2, A_1^2, A_2)$ is a solution with no vanishing subsums, then by Lemma 3 we can write

$$q_i^2 = u_0 \ell_i$$

$(i = 1, \ldots, n)$, where $(\ell_1, \ldots, \ell_n)$ comes from a finite set of cardinality bounded in terms of n and s, and u_0 is an S-unit.

Obviously, the squarefree parts of $\ell_1, \ldots, \ell_n$ are the same, say ℓ_0. Thus letting $r_i^2 = \ell_i / \ell_0$ $(i = 1, \ldots, n)$ and $u^2 = u_0 \ell_0$, we have

$$q_i = \pm u r_i$$

$(i = 1, \ldots, n)$ and we are done in this case.

Proof of Theorem 3

Hence we may assume that $(q_1^2, \ldots, q_n^2, A_1^2, A_2)$ contains a vanishing subsum. Since $q_i^2 > 0$ $(1 \leq i \leq n)$, the only possibility is that (after re-indexing $q_1, \ldots, q_n$ if necessary) we have

$$q_1^2 + \cdots + q_k^2 - A_1^2 = 0, \tag{12}$$

$$q_{k+1}^2 + \cdots + q_n^2 + 2A_2 = 0 \tag{13}$$

for some k with $1 \leq k < n$. It is easy to see that (12) and (13) do not have a vanishing subsum.

Proof of Theorem 3

Thus, similarly as above, Lemma 3 yields that

$$\begin{aligned}(q_1, \ldots, q_k) &= u(w_1, \ldots, w_k),\\ (q_{k+1}, \ldots, q_n) &= v(r_1, \ldots, r_\ell),\\ A_1 &= ut_1 \neq 0,\\ A_2 &= v^2 t_2 \neq 0,\end{aligned}$$

where $\ell = n - k$ and both $(w_1, \ldots, w_k, t_1)$ and $(r_1, \ldots, r_\ell, t_2)$ come from finite sets of S-units of cardinalities bounded in terms of n and s, and u, v are S-units.

Proof of Theorem 3

Hence (9) for $j = 1$ yields that

$$u(w_1 + \cdots + w_k) + v(r_1 + \cdots + r_\ell) = ut_1. \qquad (14)$$

If $r_1 + \cdots + r_\ell \neq 0$ then the S-unit v/u comes from a set of cardinality bounded in terms of n and s, and we are in case (i). So we may suppose that

$$\begin{aligned} w_1 + \cdots + w_k &= t_1, \\ r_1 + \cdots + r_\ell &= 0. \end{aligned}$$

As we have $k \geq 1$, $\ell \geq 1$ and, by (12) and (13),

$$\begin{aligned} w_1^2 + \cdots + w_k^2 - t_1^2 &= 0, \\ r_1^2 + \cdots + r_\ell^2 + 2t_2 &= 0, \end{aligned}$$

we obtain

$$\sigma_2(w_1, \ldots, w_k) = 0, \quad \sigma_2(r_1, \ldots, r_\ell) = t_2.$$

Proof of Theorem 3

In the next session we proved by contradiction that $k = 1$. Further, this yields $\ell = n - 1$.

In the penultimate step we may assume that

$$\begin{cases} \sigma_j(r_1, \dots, r_\ell) = A_j / v^j \neq 0 & \text{for } j \text{ even}, \\ \sigma_j(r_1, \dots, r_\ell) = 0 & \text{for } j \text{ odd}. \end{cases}$$

In particular, since $\sigma_\ell(r_1, \dots, r_\ell) = r_1 \cdots r_\ell$ cannot be zero, ℓ is even whence $n = \ell + 1$ is odd. Observing that $(x + r_1) \cdots (x + r_\ell)$ is an even polynomial, writing $\ell = 2t$ and re-indexing the S-units r_i $(1 \leq i \leq \ell)$ such that $r_{t+i} = -r_i$ $(1 \leq i \leq t)$, we see that we are in case (ii).

Proof of Theorem 3

Finally, we show that the possibilities (i) and (ii) cannot be excluded.
Indeed, if $r_1, \ldots, r_n$ is a set of rational roots of an S-polynomial of degree n, then clearly, the same is true for $ur_1, \ldots, ur_n$ for any S-unit u, showing the necessity of (i).

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On the other hand, let $r_1^2, \ldots, r_t^2$ be the rational roots of the S-polynomial $(x - r_1^2) \cdots (x - r_t^2)$. Then in the polynomial

$$(x^2 - r_1^2) \cdots (x^2 - r_t^2),$$

all the coefficients of the even powers of x are S-units (while the coefficients of the odd powers of x equal 0). Thus for any S-units u, v, all the coefficients of the polynomial

$$(x + u)(x - vr_1)(x + vr_1) \cdots (x - vr_t)(x + vr_t)$$

are S-units. This shows that (ii) cannot be excluded either.

Open Problems - 1

We wonder whether the following statement is correct:

Problem 1. *Is it true that for any primes p and q there exists an $n_1 = n_1(p, q)$ such that every polynomial $f(x) \in \mathbb{Z}[x]$ with only rational roots of which no coefficient is divisible by p or q has degree at most n_1?*

Theorem 1 shows that the answer is 'yes' for the pair of primes $(p, q) = (2, 3)$.

Open Problems - 2

A weaker statement is a restriction to S-polynomials.

Problem 2. *Is it true that for any finite set S of primes there exists an $n_2 = n_2(S)$ such that every S-polynomial $f(x) \in \mathbb{Z}[x]$ with only rational roots has degree at most n_2?*

Theorem 2 yields an affirmative answer for sets S of primes with $2, 3 \notin S$.

Open Problems - 3

The last problem is raised by Lemmas 1 and 2.

Problem 3. *Is it true that for every prime p there exists a constant $c(p)$ such that if $f(x) \in \mathbb{Z}[x]$ has only rational roots and none of the coefficients of f is divisible by p, then $\deg(f) + 1$ in its p-adic expansion has at most $c(p)$ non-zero digits? In particular, can one take $c(p) = p - 1$?*

Lemmas 1 and 2 show that the answer is 'yes' with $c(p) = p - 1$ for $p = 2, 3$. Note that an affirmative answer to Problem 3 through a deep result of Stewart (1980, J. reine angew. Math.) would yield positive answers to Problems 1 and 2, as well.

- L. Hajdu - R. Tijdeman - N. Varga: *On polynomials with only rational roots.* Mathematika (submitted)