# *Number of solutions to a special type of unit equations in two unknowns III*

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## Plan of Talk

1 Main equation

2 Motivation - review of Part II

3 Results

4 Idea for proofs

5 Open problems

**purely exponential Diophantine equation**

$$a^x + b^y = c^z$$

$a, b, c$ given positive integers $> 1$
coprime

$x, y, z$ positive integer unknowns

- $3^x + 10^y = 13^z$
- $\boxed{\text{unknown} = 1}$ allowed

## Basic facts

· $\#\{\,(x,y,z)\,\} \leq$ absolute constant $\leq 2^{32}$.

$\leftarrow$-- Schmidt Subspace Theorem

· $x, y, z < \mathcal{C}_{\text{eff}}(a, b, c)$.

$\leftarrow$-- Baker's theory OR its $p$-adic form

In recent years, there has been important progress on estimating number of solutions.

**Conjecture [Bennett, '01]**

*For any* $a, b, c \in \mathbb{N}$ ; $a, b > 1$, $\gcd(a, b) = 1$,

$a, b$ *not perfect powers,*

*there is* <u>*at most* 1</u> *sol to*

$$a^x - b^y = c \quad x, y \geq 1,$$

*except for* $(a, b, c) = (2, 3, 5), (2, 3, 13), (2, 5, 3),$

$(3, 2, 1), (13, 3, 10)$ *or* $(91, 2, 89)$.

## Exceptional cases

$$2^3 - 3 = 2^5 - 3^3 = 5$$

$$2^4 - 3 = 2^8 - 3^5 = 13$$

$$2^3 - 5 = 2^7 - 5^3 = 3$$

$$3 - 2 = 3^2 - 2^3 = 1$$

$$13 - 3 = 13^3 - 3^7 = 10$$

$$91 - 2 = 91^2 - 2^{13} = 89$$

[Bennett, '01] confirmed his conj for

- $c \geq b^{2a^2 \log a}$ ;
- $c \leq b^y/6000$ or $c \leq 100$ ;
- $b \equiv \pm 1 \pmod{a}$ with $a$ prime.

※ $a$ : base of the *greatest term* in $a^x - b^y = c$

($\rightsquigarrow$ $a$ : Fermat primes $\Rightarrow$ Conj)

Motivation of Part II

**3-variable version** of the 3rd result above

# 3-variable version of Bennett's conj

**Conjecture [Scott & Styer, '16]** *atmost1*

*For any* $a, b, c \in \mathbb{N}_{>1}$ ; $\gcd(a, b, c) = 1$,

$a < b$, $a, b, c$ *not perfect powers,*

*there is at most 1 sol to*

$$a^x + b^y = c^z \quad x, y, z \geq 1,$$

*except for* $(a, b, c) = (3,5,2), (3,13,2), (2,5,3),$
$(2,7,3), (2,3,11), (3,10,13), (2,3,35), (2,89,91),$
$(2,5,133), (2,3,259), (3,13,2200), (2,91,8283)$ *or*
$(2, 2^r-1, 2^r+1)$ ; $r = 2, 4, 5, \cdots$.

## Exceptional cases

$3 + 5 = 2^3 \quad 3^3 + 5 = 2^5 \quad 3 + 5^3 = 2^7$

$3 + 13 = 2^4 \quad 3^5 + 13 = 2^8$

$2^2 + 5 = 3^2 \quad 2 + 5^2 = 3^3$

$2 + 7 = 3^2 \quad 2^5 + 7^2 = 3^4$

$2^3 + 3 = 11 \quad 2 + 3^2 = 11$

$3 + 10 = 13 \quad 3^7 + 10 = 13^3$

$2^5 + 3 = 35 \quad 2^3 + 3^3 = 35$

$$3+5=2^3 \quad 3^3+5=2^5$$

$$2+89=91 \quad 2^{13}+89=91^2$$

$$2^7+5=133 \quad 2^3+5^3=133$$

$$2^8+3=259 \quad 2^4+3^5=259$$

$$3^7+13=2200 \quad 3+13^3=2200$$

$$2^{13}+91=8283 \quad 2+91^2=8283$$

$$2+(2^r-1)=2^r+1 \quad 2^{r+2}+(2^r-1)^2=(2^r+1)^2$$

# From Part II (2024)

**Proposition**

$$a \equiv \pm 1 \mod c \quad \text{or} \quad b \equiv \pm 1 \mod c$$

$$\Rightarrow \ \textit{atmost1}$$

**Corollary**

$$c \in \{2, 3, 6\} \Rightarrow \textit{atmost1}$$

- 『 $p \nmid \mathcal{A} \Rightarrow \mathcal{A} \equiv \pm 1 \mod p$ 』 for $p \in \{2, 3\}$
- **another proof of [Scott, '93] for $c = 2$**

# Corollaries of Part II

*For any fixed* $c$ *with*

$$2^{\nu_2(c)} > \sqrt{c} \quad \textit{or} \quad 3^{\nu_3(c)} > \sqrt{c},$$

*atmost1 is true,*

*except for only finitely many* $(a, b)$.

$c = p^n \cdot k \Rightarrow$ *atmost1*

*where* $p \in \{2, 3\}$, $k \not\equiv 0 \pmod{p}$, $n \geq n_{\text{eff}}(k)$.

$\rightsquigarrow$ atmost1 is true for $\infty$ **many** values of $c$.

**For other values of $c$**

**can we prove atmost1 conj**

**except for only finitely many $(a, b)$ ?**

## Main Problem

Fix the value of $c$, and work out the following:

[1] Proving $N \leq 1$, except for only finitely many $(a, b)$.

$N(a, b, c)$ : number of sols to $a^x + b^y = c^z$

[2] Finding a way to enumerate all possible $(a, b)$ described as exceptional in [1] in a *concrete* finite time.

[3] Sieving all $(a, b)$ found in [2] completely.

—

[1] *ineffective* [2] *effective* [3] *complete*

## Best of our knowledge

- $c = 2$ *complete* by [Scott, '93]

- $c = 2, 3, 6$ *complete* by Part II

- $2^{\nu_2(c)} > \sqrt{c}$ or $3^{\nu_3(c)} > \sqrt{c}$ *effective* by Part II

- $c = 2^{\text{large}} \cdot k,\ 3^{\text{large}} \cdot k$ *complete* by Part II

- $c = \underbrace{5, 17, 257, 65537}_{\text{Fermat primes}}$ *complete* by Part II

# Status of atmost1 conj for small $c$

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $c$ | **2** | **3** | **5** | 6 | 7 | 10 | **11** | 12 |
| status | ✓ | ✓ | ✓ | ✓ | ? | ? | ? | ✓* |
| **13** | 14 | 15 | **17** | 18 | 19 | 20 | 21 | 22 |
| ? | ? | ? | ✓ | ✓* | ? | ? | ? | ? |
| 23 | 24 | 26 | 28 | 29 | 30 | 31 | **33** | 34 |
| ? | ✓* | ? | ? | ? | ? | ? | ? | ? |
| **35** | 37 | 38 | 39 | 40 | 41 | 42 | 43 | 44 |
| ? | ? | ? | ? | ✓* | ? | ? | ? | ? |

✓ complete ✓* effective

**Theorem 1**

*For any prime* $c$ *of the form* $3 \cdot 2^r + 1$ *with some* $r \in \mathbb{N}$,

$$N(a,b,c) \leq 1,$$

*except for only finitely many* $(a,b)$.

- $c = 7, 13, 97, 193, 769, 12289, 786433, \cdots$
- *ineffective* for each $c$
- $^{\exists}$ condition on $c$ to make Th1 **effective**

**Theorem 2**

$N(a, b, 13) \leq 1$,

*except for* $(a, b) = (3, 10)$ *or* $(10, 3)$.

- $N(3, 10, 13) = 2$
- *complete* for $c = 13$

## Status of atmost1 conj for small $c$

| $c$ | **2** | **3** | **5** | 6 | 7 | 10 | **11** | 12 |
|---|---|---|---|---|---|---|---|---|
| status | ✓ | ✓ | ✓ | ✓ | ✓ | ? | ? | ✓* |
| **13** | 14 | 15 | **17** | 18 | 19 | 20 | 21 | 22 |
| ✓ | ? | ? | ✓ | ✓* | ? | ? | ? | ? |
| 23 | 24 | 26 | 28 | 29 | 30 | 31 | **33** | 34 |
| ? | ✓* | ? | ? | ? | ? | ? | ? | ? |
| **35** | 37 | 38 | 39 | 40 | 41 | 42 | 43 | 44 |
| ? | ? | ? | ? | ✓* | ? | ? | ? | ? |

✓ complete ✓* effective ✓ ineffective

Notation

- For $M > 1$ & $h$ with $\gcd(h, M) = 1$,

  $e_M(h)$ : least $e \geq 1$ s.t. $h^e \equiv \pm 1 \mod M$

- For $M > 1$ & $h \neq 0$,

  $\nu_M(h)$ : greatest $\nu \geq 0$ s.t. $M^\nu \mid h$

Remarks

We may assume

$$e_{c'}(a) = e_{c'}(b) \quad := E,$$

where $c' > 2$ is any fixed divisor of $c$.

Part II contributed to $\boxed{E = 1}$ with $c' = c$

(or, $E$ : even with $c$ prime).

$$a, b, c \in \mathbb{N}_{>1}$$

$$c' \mid c \; ; \; c' > 2, \; \mathrm{gcd}(c', \varphi(c')) = 1$$

$$e_{c'}(a) = e_{c'}(b) \quad := E$$

$\mathcal{C}_i = \mathcal{C}_i(c)$ effectively computable

**Lemma 1**

*If* $max\{a,b\} > \mathcal{C}_1,$ *then*

$$\text{gcd}(x,y,c) = 1$$

*for any sol* $(x,y,z)$ *to* $a^x + b^y = c^z$.

[Proof]

· elementary for prime $c$

· For composite $c$, using lower bound for

$$P(\text{ integer}^m + \text{integer}^m).$$

($\rightsquigarrow \mathcal{C}_1$ huge)

**Lemma 2**

*If* $max\{a,b\} > \mathcal{C}_1$, *then*

$$z \ll_c \log a \log b$$

*for any sol* $(x,y,z)$ *to* $a^x + b^y = c^z$.

[Proof]

With Lem 1 & gcd( $c'$ , $\varphi(c')$ ) $= 1$,

using upper bound for

$$\nu_{c'}(a^x + b^y) \quad = \nu_{c'}(c^z) \geq z$$

by $c'$-adic Baker of [Bugeaud, '02].

$N(a,b,c) > 1$

$a^x + b^y = c^z \quad a^X + b^Y = c^Z \quad z \leq Z$

$\rightsquigarrow \; c'^z \;\Big|\; \mathrm{gcd}(\, a^E \pm 1,\; b^E \pm 1\,) \cdot \Delta$

$$\Delta := |\, xY - Xy \,| \quad > 0$$

$\rightsquigarrow$ $a, b$ : close* to 1 $c'$-adically

$\because$ $\Delta, E$ small ←-- $\Delta \ll_c z$ by Lem 2 & $E \mid \varphi(c')$

$$\mathcal{D} := \frac{c'^z}{\mathrm{gcd}(\, c'^z, \Delta\,)} \;\Big|\; \mathrm{gcd}(a^E \pm 1,\; b^E \pm 1) \qquad \cdots \text{☆}$$

## Lemma 3

*If* $max\{a, b\} > \mathcal{C}_2,$ *then*

$$z \cdot Z \ll_c \log a \log b.$$

[Proof]

Based on ☆, with $\gcd(c', \varphi(c')) = 1$,

using upper bound for

$$\nu_{\mathcal{D}}(a^X + b^Y) \quad = \nu_{\mathcal{D}}(c^Z) \geq Z/z$$

by $\mathcal{D}$-adic Baker of [Bugeaud, '02].

**Lemma 4**

*If* $max\{a,b\} > \mathcal{C}_2,$ *then*

$$x,\ \ y,\ \ X,\ \ Y \ll_c\ 1.$$

Remark

$$\begin{cases} A^x + B^y = (\text{fix})^z \\ A^X + B^Y = (\text{fix})^Z \\ (x,y,z) \neq (X,Y,Z) \end{cases} \quad \Rightarrow \quad x,y,X,Y:\ \text{finite}$$

**Lemma 5**

*If* $max\{a, b\} > \mathcal{C}_3$, *then*

$$min\{x, y\} = 1, \ min\{X, Y\} = 1.$$

[Proof]

- lower bound for $P(\text{int}^m + \text{int}^n)$ ($\rightsquigarrow \mathcal{C}_3$ **huge**)
- $\pi$-adic Baker in $\mathbb{Q}(i)$ of [Bugeaud & Laurent, '96]

Remark $x = X$ or $y = Y \Rightarrow a, b < \mathcal{C}_{\text{ineff}}(c)$.

$\because \ x = X$

$\Rightarrow \ b^Y - b^y = c^Z - c^z \ ; \ \ y, Y \ll_c 1$

$\Rightarrow \ b, z, Z(, a) < \mathcal{C}(c)$ by [Bugeaud & Luca, '06].

**Lemma 6**

*If* $c$ *: prime* $(\rightsquigarrow E \geq 3$, *by* Part II$)$,

*and* $max\{a, b\} > \mathcal{C}_4$, *then*

$$max\{x, y\} \leq E - 2.$$

※ $max\{x, y\} \leq \left\lfloor \frac{E \log c}{\log c'} \right\rfloor$ for composite $c$.

[Proof]

- $c^z \ll min\{a, b\}^{E-1}$ ←-- ☆ & $min\{a, b\} < c^{z/max\{a,b\}}$
- lower bound of $P(f(\text{int}))$ for $f \in \mathbb{Z}[t]$

  $(\rightsquigarrow \mathcal{C}_4$ : **huge**$)$

[Proof of Th1]

$c = 3 \cdot 2^r + 1$ : prime

$E > 1, \ 2 \nmid E$ by Part II

$E \mid \varphi(c) \ = c - 1 = 3 \cdot 2^r$

$\therefore \ E = 3$

$N(a, b, c) > 1$

If $max\{a, b\} > max\{\mathcal{C}_1, \cdots, \mathcal{C}_4\}$, then

$$\begin{cases} min\{X, Y\} = 1 \\ max\{x, y\} \leq E - 2 \ = 1 \end{cases}$$

$\therefore \ a, b < \mathcal{C}(c)$ ■

# Idea for Proof of Th2

Q1 For each $c$ in

$$c = 7, 13, 97, 193, 769, 12289, 786433, \cdots$$

how we make Th1 effective?

A It is enough to

*find positive numbers* $\epsilon, C$ *with* $\epsilon < 0.6$ *s.t.*

$$\left| \sqrt{c} - \frac{p}{q} \right| > \frac{C}{q^{1+\epsilon}}$$

*holds for all* $p, q \in \mathbb{N}$ *s.t.* $q$ *equals a power of* $c$.

※ [Bauer-Bennett, '01] solves this *only* for $c = 13$ with $\epsilon = 0.53$.

Q2 For $c = 13$, how we avoid to rely on $\mathcal{C}_3$ ?

A Instead of $P(\text{int}^m + \text{int}^n)$, we use

- Parity lemma of [Scott, '93];
- result of [Bennett-Siksek, '23] on

$$S^2 - 13^k = T^n \quad 13 \nmid S,\ k \geq 1,\ n \geq 3.$$

Q3 For $c = 13$, how we avoid to rely on $\mathcal{C}_4$ ?

A Instead of $P(f(\text{int}))$, we use

$\pi$-adic Baker in $\mathbb{Q}(\omega)$ of [Bugeaud & Laurent, '96].

## Open problems

Q For any $x$ and $Y$ with

$2 \le x \le 4,\ 2 \le Y \le 3.74 \cdot 10^{11},\ xY \equiv 1 \mod 5,$

prove that there are only finitely many $(a, b)$ s.t.

$$a^x + b = 11^z, \quad a + b^Y = 11^Z$$

for some $z$ and $Z$ with $z \le Z$.

Q For any $x$ and $Y$ with

$x \in \{2,4,5,7,8\},\ 2 \leq Y \leq 1.3 \cdot 10^{13},\ xY \equiv 1 \mod 3,$

prove that there are only finitely many $(a,b)$ s.t.

$$a^x + b = 19^z, \quad a + b^Y = 19^Z$$

for some $z$ and $Z$ with $z \leq Z$.

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*Thank you very much*
*for your attention!*