The sup-norm problem of automorphic forms: spherical and non-spherical

Péter Maga[1]

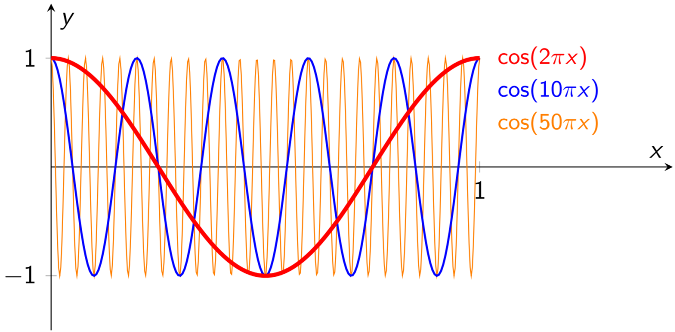
Rényi Institute, Budapest

Online Number Theory Seminar
Debrecen
2026 February 13

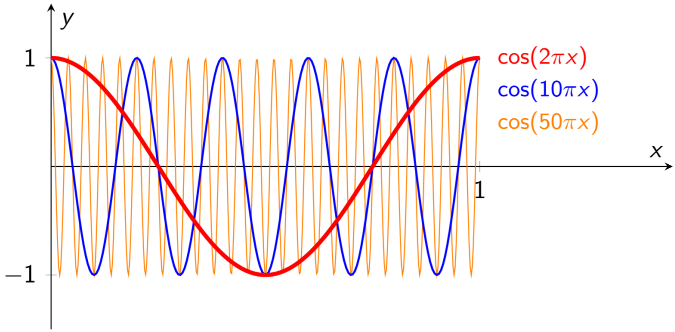
[1] based on an ongoing work with Valentin Blomer, Gergely Harcos and Djordje Milićević

Warm-up

Warm-up

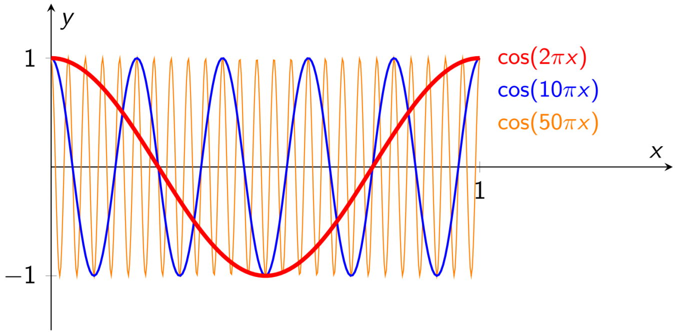


Warm-up



The L^2-normalized periodic solutions of the differential equation $\frac{d^2}{dx^2} f(x) = c \cdot f(x)$ remain bounded, even when c gets large.

Warm-up



The L^2-normalized periodic solutions of the differential equation $\frac{d^2}{dx^2} f(x) = c \cdot f(x)$ remain bounded, even when c gets large.

Fundamental question (in the physics language): how is the mass of high-energy eigenstates distributed?

Classical setup I

Classical setup I

We recall Poincaré's model for (the two-dimensional) hyperbolic geometry:

Classical setup I

We recall Poincaré's model for (the two-dimensional) hyperbolic geometry:

$$\mathcal{H} := \{z = x + iy \in \mathbb{C} : \Im z = y > 0\}, \quad \mathrm{dist}(P, Q) := \inf_{P \sim^{\gamma} Q} \int_{\gamma} \frac{|dz|}{\Im z}.$$

Classical setup I

We recall Poincaré's model for (the two-dimensional) hyperbolic geometry:

$$\mathcal{H} := \{z = x+iy \in \mathbb{C} : \Im z = y > 0\}, \quad \mathrm{dist}(P, Q) := \inf_{P \sim^{\gamma} Q} \int_{\gamma} \frac{|dz|}{\Im z}.$$

Its group of isometries is $\mathrm{(P)SL}_2(\mathbb{R})$ with the action

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}(z) := \frac{az+b}{cz+d}.$$

Classical setup I

We recall Poincaré's model for (the two-dimensional) hyperbolic geometry:

$$\mathcal{H} := \{z = x+iy \in \mathbb{C} : \Im z = y > 0\}, \quad \mathrm{dist}(P, Q) := \inf_{P \sim^\gamma Q} \int_\gamma \frac{|dz|}{\Im z}.$$

Its group of isometries is $(\mathrm{P})\mathrm{SL}_2(\mathbb{R})$ with the action

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} (z) := \frac{az+b}{cz+d}.$$

The invariant measure (area) is given by $y^{-2}\, dx\, dy$.

Classical setup I

We recall Poincaré's model for (the two-dimensional) hyperbolic geometry:

$$\mathcal{H} := \{z = x+iy \in \mathbb{C} : \Im z = y > 0\}, \quad \operatorname{dist}(P, Q) := \inf_{P \sim^\gamma Q} \int_\gamma \frac{|dz|}{\Im z}.$$

Its group of isometries is $(\mathrm{P})\mathrm{SL}_2(\mathbb{R})$ with the action

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}(z) := \frac{az+b}{cz+d}.$$

The invariant measure (area) is given by $y^{-2}\,dx\,dy$.
The Laplacian is

$$\Delta := -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right).$$

Classical setup II

Classical setup II

We take a discrete subgroup defined arithmetically: $\Gamma := \mathrm{SL}_2(\mathbb{Z})$.

Classical setup II

We take a discrete subgroup defined arithmetically: $\Gamma := \mathrm{SL}_2(\mathbb{Z})$. Then the modular surface is defined as the quotient $X := \Gamma \backslash \mathcal{H}$.

Classical setup II

We take a discrete subgroup defined arithmetically: $\Gamma := \mathrm{SL}_2(\mathbb{Z})$. Then the modular surface is defined as the quotient $X := \Gamma\backslash\mathcal{H}$. And we consider some very special functions on this space: those which are

Classical setup II

We take a discrete subgroup defined arithmetically: $\Gamma := \mathrm{SL}_2(\mathbb{Z})$. Then the modular surface is defined as the quotient $X := \Gamma\backslash\mathcal{H}$. And we consider some very special functions on this space: those which are

- square-integrable (and assume they are already L^2-normalized),

Classical setup II

We take a discrete subgroup defined arithmetically: $\Gamma := \mathrm{SL}_2(\mathbb{Z})$. Then the modular surface is defined as the quotient $X := \Gamma\backslash\mathcal{H}$. And we consider some very special functions on this space: those which are

- square-integrable (and assume they are already L^2-normalized),
- eigenfunctions of Δ and the Hecke operators,

Classical setup II

We take a discrete subgroup defined arithmetically: $\Gamma := \mathrm{SL}_2(\mathbb{Z})$. Then the modular surface is defined as the quotient $X := \Gamma\backslash\mathcal{H}$. And we consider some very special functions on this space: those which are

- square-integrable (and assume they are already L^2-normalized),
- eigenfunctions of Δ and the Hecke operators,
- cuspidal.

The sup-norm problem in the classical setup

The sup-norm problem in the classical setup

Question

For such a ϕ, what can be said about

$$\sup_{z \in X} |\phi(z)|?$$

The sup-norm problem in the classical setup

Question

For such a ϕ, what can be said about

$$\sup_{z \in X} |\phi(z)|?$$

Theorem (Sarnak)

If $X = \Gamma \backslash S$ is a compact, locally symmetric space, then we have, for the eigenforms f of the invariant differential operators,

$$\sup_{z \in X} |\phi(z)| \ll_X \lambda_\phi^{\frac{\dim - \mathrm{rk}}{4}}.$$

Comments and the number-theoretic goal

Comments and the number-theoretic goal

The exponent $(\dim - \mathrm{rk})/4$ is sharp in this generality, for example, eigenfunctions of the Laplacian on the sphere S^2 grow with this rate.

Comments and the number-theoretic goal

The exponent $(\dim - \mathrm{rk})/4$ is sharp in this generality, for example, eigenfunctions of the Laplacian on the sphere S^2 grow with this rate.

If X is not compact (many cases in number theory, e.g. $\Gamma\backslash\mathcal{H}$ above), then the bound is still true restricted to compacta, $\Omega \subset X$,

$$\sup_{z\in\Omega} |\phi(z)| \ll_\Omega \lambda_\phi^{\frac{\dim - \mathrm{rk}}{4}}.$$

Comments and the number-theoretic goal

The exponent $(\dim - \mathrm{rk})/4$ is sharp in this generality, for example, eigenfunctions of the Laplacian on the sphere S^2 grow with this rate.
If X is not compact (many cases in number theory, e.g. $\Gamma\backslash\mathcal{H}$ above), then the bound is still true restricted to compacta, $\Omega \subset X$,

$$\sup_{z\in\Omega} |\phi(z)| \ll_\Omega \lambda_\phi^{\frac{\dim - \mathrm{rk}}{4}}.$$

The arithmetic sup-norm problem

Can we save over Sarnak's bound if X further has arithmetic symmetries (Hecke operators) and ϕ is assumed to be an eigenfunction of these Hecke operators, too, i.e. can we write, with some $\delta > 0$ (which might depend on X),

$$\sup_{z\in\Omega} |\phi(z)| \ll_\Omega \lambda_\phi^{\frac{\dim - \mathrm{rk}}{4} - \delta}?$$

Some results on the group $\mathrm{GL}(2)$

Some results on the group $\mathrm{GL}(2)$

Theorem (Iwaniec–Sarnak, 1995)

In the presented setup,

$$\sup_{z \in X} |\phi(z)| \ll_\varepsilon \lambda_\phi^{5/24+\varepsilon},$$

a saving over $(\dim - \mathrm{rk})/4 = (2-1)/4 = 1/4 = 6/24$.

Some results on the group $\mathrm{GL}(2)$

Theorem (Iwaniec–Sarnak, 1995)

In the presented setup,

$$\sup_{z \in X} |\phi(z)| \ll_\varepsilon \lambda_\phi^{5/24+\varepsilon},$$

a saving over $(\dim - \mathrm{rk})/4 = (2-1)/4 = 1/4 = 6/24$.

Theorem (Blomer–Harcos–Milićević, 2016)

For spherical Hecke–Maaß cusp forms ϕ on $\mathrm{GL}_2(\mathbb{Z}[i]) \backslash \mathrm{GL}_2(\mathbb{C})$ of trivial central character,

$$\sup |\phi| \ll_\varepsilon \lambda_\phi^{5/12+\varepsilon}.$$

a saving over $(\dim - \mathrm{rk})/4 = (3-1)/4 = 1/2 = 6/12$.

Some results on the group $\mathrm{GL}(2)$

Theorem (Iwaniec–Sarnak, 1995)

In the presented setup,

$$\sup_{z\in X} |\phi(z)| \ll_\varepsilon \lambda_\phi^{5/24+\varepsilon},$$

a saving over $(\dim - \mathrm{rk})/4 = (2-1)/4 = 1/4 = 6/24$.

Theorem (Blomer–Harcos–Milićević, 2016)

For spherical Hecke–Maaß cusp forms ϕ on $\mathrm{GL}_2(\mathbb{Z}[i])\backslash\mathrm{GL}_2(\mathbb{C})$ of trivial central character,

$$\sup |\phi| \ll_\varepsilon \lambda_\phi^{5/12+\varepsilon}.$$

a saving over $(\dim - \mathrm{rk})/4 = (3-1)/4 = 1/2 = 6/12$.

Theorem (Blomer–Harcos–M.–Milićević, 2020)

Over any number field. (Same exponents over totally real and CM fields, weaker saving in other cases.)

Applications, relations

Applications, relations

Studying the sup-norm is related to many other things in the analytic theory of automorphic forms, e.g.

Applications, relations

Studying the sup-norm is related to many other things in the analytic theory of automorphic forms, e.g.

- subconvexity of L-functions,

Applications, relations

Studying the sup-norm is related to many other things in the analytic theory of automorphic forms, e.g.

- subconvexity of L-functions,
- zero sets of automorphic forms (Ghosh–Reznikov–Sarnak, Rudnick),

Applications, relations

Studying the sup-norm is related to many other things in the analytic theory of automorphic forms, e.g.

- subconvexity of L-functions,
- zero sets of automorphic forms (Ghosh–Reznikov–Sarnak, Rudnick),
- averaged versions of the hyperbolic circle problem (Petridis–Risager),

Applications, relations

Studying the sup-norm is related to many other things in the analytic theory of automorphic forms, e.g.

- subconvexity of L-functions,
- zero sets of automorphic forms (Ghosh–Reznikov–Sarnak, Rudnick),
- averaged versions of the hyperbolic circle problem (Petridis–Risager),
- etc.

The $\mathrm{GL}_n(\mathbb{R})$ setup I: spaces and functions

The $\mathrm{GL}_n(\mathbb{R})$ setup I: spaces and functions

We introduce

$$\begin{aligned} G &:= \mathrm{PGL}_n(\mathbb{R}), && \text{a Lie group,} \\ \Gamma &:= \mathrm{PGL}_n(\mathbb{Z}), && \text{a discrete subgroup,} \\ K &:= \mathrm{PO}_n(\mathbb{R}), && \text{a maximal compact subgroup.} \end{aligned}$$

The $\mathrm{GL}_n(\mathbb{R})$ setup I: spaces and functions

We introduce

$$G := \mathrm{PGL}_n(\mathbb{R}), \qquad \text{a Lie group,}$$
$$\Gamma := \mathrm{PGL}_n(\mathbb{Z}), \qquad \text{a discrete subgroup,}$$
$$K := \mathrm{PO}_n(\mathbb{R}), \qquad \text{a maximal compact subgroup.}$$

Set

$$X := \Gamma \backslash G / K,$$

and we consider functions $f : X \to \mathbb{C}$, often as functions on $\mathrm{GL}_n(\mathbb{R})$ such that

$$f(z\gamma gk) = f(g)$$

for any $z \in Z(\tilde{G})$, $\gamma \in \tilde{\Gamma}$, $g \in \tilde{G}$, $k \in \tilde{K}$, where $\tilde{\cdot}$ is $\cdot$ without the projectivization.

The $\mathrm{GL}_n(\mathbb{R})$ setup II: differential operators

The $\mathrm{GL}_n(\mathbb{R})$ setup II: differential operators

Set $\mathfrak{g}$ for $n \times n$ matrices of 0 trace, this is the Lie algebra of G. Its elements act on smooth functions on G as

$$(Xf)(g) := \frac{d}{dt} f(g \exp(tX))\Big|_{t=0}.$$

The $\mathrm{GL}_n(\mathbb{R})$ setup II: differential operators

Set $\mathfrak{g}$ for $n \times n$ matrices of 0 trace, this is the Lie algebra of G. Its elements act on smooth functions on G as

$$(Xf)(g) := \frac{d}{dt} f(g \exp(tX))\Big|_{t=0}.$$

We complexify it as $\mathfrak{g}_{\mathbb{C}}$, and consider the universal enveloping algebra $U(\mathfrak{g}_{\mathbb{C}})$. These are the higher-order differentiations (with complex coefficients).

The $\mathrm{GL}_n(\mathbb{R})$ setup II: differential operators

Set $\mathfrak{g}$ for $n \times n$ matrices of 0 trace, this is the Lie algebra of G. Its elements act on smooth functions on G as

$$(Xf)(g) := \frac{d}{dt} f(g \exp(tX)) \Big|_{t=0}.$$

We complexify it as $\mathfrak{g}_{\mathbb{C}}$, and consider the universal enveloping algebra $U(\mathfrak{g}_{\mathbb{C}})$. These are the higher-order differentiations (with complex coefficients).
We consider then its center $\mathcal{D} := Z(U(\mathfrak{g}_{\mathbb{C}}))$, these are the higher-order differentiations (with complex coefficients) which commute with the right action of G. Why do we need the machinery? Typically $\mathfrak{g}$-derivatives of a function on X is not a function on X (right-K-invariance is lost), but $\mathcal{D}$-derivatives of a function on X is a function on X, because the $\mathcal{D}$-action commutes with the K-action.

The $\mathrm{GL}_n(\mathbb{R})$ setup III: Hecke operators

The $\mathrm{GL}_n(\mathbb{R})$ setup III: Hecke operators

Let p be a prime number, and $\mathbf{a} := (a_1, \ldots, a_n)$ is an n-tuple of integers, where $a_1 \geq \ldots \geq a_n \geq 0$. We can write

$$\tilde{\Gamma} \operatorname{diag}(p^{a_1}, \ldots, p^{a_n}) \tilde{\Gamma} = \bigcup_{j \in J} \tilde{\Gamma} M_j$$

with a finite set J.

The $\mathrm{GL}_n(\mathbb{R})$ setup III: Hecke operators

Let p be a prime number, and $\mathbf{a} := (a_1, \ldots, a_n)$ is an n-tuple of integers, where $a_1 \geq \ldots \geq a_n \geq 0$. We can write

$$\tilde{\Gamma}\,\mathrm{diag}(p^{a_1}, \ldots, p^{a_n})\tilde{\Gamma} = \bigcup_{j \in J} \tilde{\Gamma} M_j$$

with a finite set J.
For any $f : X \to \mathbb{C}$, the unnormalized Hecke operator to this data is defined as

$$(T_{p,\mathbf{a}} f)(g) := \sum_{j \in J} f(M_j g).$$

It is a function on X.

The $\mathrm{GL}_n(\mathbb{R})$ setup IV: cuspidality

The $\mathrm{GL}_n(\mathbb{R})$ setup IV: cuspidality

For any $n = n_1 + \ldots + n_k$ with $k \geq 2$, $n_1, \ldots, n_k > 0$, we consider the unipotent subgroup

$$U_{n_1,\ldots,n_k} := \left\{ \begin{pmatrix} I_{n_1} & * & \ldots & * \\ 0 & I_{n_2} & \ldots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & I_{n_k} \end{pmatrix} \right\},$$

where I_m is the $m \times m$ identity matrix.

The $\mathrm{GL}_n(\mathbb{R})$ setup IV: cuspidality

For any $n = n_1 + \ldots + n_k$ with $k \geq 2$, $n_1, \ldots, n_k > 0$, we consider the unipotent subgroup

$$U_{n_1,\ldots,n_k} := \left\{ \begin{pmatrix} I_{n_1} & * & \ldots & * \\ 0 & I_{n_2} & \ldots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & I_{n_k} \end{pmatrix} \right\},$$

where I_m is the $m \times m$ identity matrix. A function $f : X \to \mathbb{C}$ is said to be cuspidal, if for almost every $g \in G$,

$$\int_{U_{n_1,\ldots,n_k}(\mathbb{Z})\backslash U_{n_1,\ldots,n_k}(\mathbb{R})} f(ug)\, du = 0.$$

In fact, it suffices to assume this for $k = 2$.

Hecke–Maaß cusp forms and sup-norm bounds

Hecke–Maaß cusp forms and sup-norm bounds

Definition

A function $\phi : X \to \mathbb{C}$ is a Hecke–Maaß cusp form, if it is a joint eigenfunction of $\mathcal{D}$ and all the Hecke operators $T_{p,\mathbf{a}}$ (for any p and $\mathbf{a}$), and it is cuspidal. Note: then it is automatically in $L^2(X)$, and we will assume below all along that it is L^2-normalized. Notation: its Laplace eigenvalue will be denoted by λ_ϕ (the Laplace operator is in $\mathcal{D}$, so this makes sense).

Hecke–Maaß cusp forms and sup-norm bounds

Definition

A function $\phi : X \to \mathbb{C}$ is a Hecke–Maaß cusp form, if it is a joint eigenfunction of $\mathcal{D}$ and all the Hecke operators $T_{p,\mathbf{a}}$ (for any p and $\mathbf{a}$), and it is cuspidal. Note: then it is automatically in $L^2(X)$, and we will assume below all along that it is L^2-normalized. Notation: its Laplace eigenvalue will be denoted by λ_ϕ (the Laplace operator is in $\mathcal{D}$, so this makes sense).

By Sarnak's general bound, for any compact $\Omega \subset X$,

$$\sup_{g \in \Omega} |\phi(g)| \ll_\Omega \lambda_\phi^{\frac{n(n-1)}{8}}.$$

Hecke–Maaß cusp forms and sup-norm bounds

Definition

A function $\phi : X \to \mathbb{C}$ is a Hecke–Maaß cusp form, if it is a joint eigenfunction of $\mathcal{D}$ and all the Hecke operators $T_{p,\mathbf{a}}$ (for any p and $\mathbf{a}$), and it is cuspidal. Note: then it is automatically in $L^2(X)$, and we will assume below all along that it is L^2-normalized. Notation: its Laplace eigenvalue will be denoted by λ_ϕ (the Laplace operator is in $\mathcal{D}$, so this makes sense).

By Sarnak's general bound, for any compact $\Omega \subset X$,

$$\sup_{g \in \Omega} |\phi(g)| \ll_\Omega \lambda_\phi^{\frac{n(n-1)}{8}}.$$

Theorem (Blomer–M., 2016)

There is some $\delta > 0$ depending only on n such that

$$\sup_{g \in \Omega} |\phi(g)| \ll_\Omega \lambda_\phi^{\frac{n(n-1)}{8} - \delta}.$$

A glance at other groups

A glance at other groups

Theorem (Blomer–Pohl, 2016)

There exists some $\delta > 0$ such that for any spherical Hecke–Maaß cusp form ϕ on $\mathrm{Sp}_4(\mathbb{Z})\backslash\mathrm{Sp}_4(\mathbb{R})$, and any compact Ω,

$$\sup_{g\in\Omega} |\phi(g)| \ll_\Omega \lambda_\phi^{1-\delta}.$$

A glance at other groups

Theorem (Blomer–Pohl, 2016)

There exists some $\delta > 0$ such that for any spherical Hecke–Maaß cusp form ϕ on $\mathrm{Sp}_4(\mathbb{Z})\backslash\mathrm{Sp}_4(\mathbb{R})$, and any compact Ω,

$$\sup_{g\in\Omega} |\phi(g)| \ll_\Omega \lambda_\phi^{1-\delta}.$$

Theorem (Marshall, 2014+??, M.–Zábrádi, 2024)

There exists some $\delta > 0$ depending only on n such that for any spherical Hecke–Maaß cusp form ϕ on $\mathrm{PGL}_n(\mathbb{Z}[i])\backslash\mathrm{PGL}_n(\mathbb{C})$, and any compact Ω,

$$\sup_{g\in\Omega} |\phi(g)| \ll_\Omega \lambda_\phi^{\frac{n(n-1)}{4}-\delta}.$$

A representation-theoretic point of view

A representation-theoretic point of view

Given a Hecke–Maaß cusp form ϕ on X as above, it generates a unitary representation as follows.

A representation-theoretic point of view

Given a Hecke–Maaß cusp form ϕ on X as above, it generates a unitary representation as follows. First, we consider its right translates

$$\phi_x(g) := \phi(gx), \qquad x \in G,$$

which are not necessarily (and typically are not) functions on X but only on $\Gamma\backslash G$.

A representation-theoretic point of view

Given a Hecke–Maaß cusp form ϕ on X as above, it generates a unitary representation as follows. First, we consider its right translates

$$\phi_x(g) := \phi(gx), \qquad x \in G,$$

which are not necessarily (and typically are not) functions on X but only on $\Gamma\backslash G$. Then we consider

$$V_\phi := \text{closure of the linear span of } \{\phi_x : x \in G\},$$

where the closure is meant in $L^2(\Gamma\backslash G)$. Then V_ϕ is a unitary representation of G.

A representation-theoretic point of view

Given a Hecke–Maaß cusp form ϕ on X as above, it generates a unitary representation as follows. First, we consider its right translates

$$\phi_x(g) := \phi(gx), \qquad x \in G,$$

which are not necessarily (and typically are not) functions on X but only on $\Gamma\backslash G$. Then we consider

$$V_\phi := \text{closure of the linear span of } \{\phi_x : x \in G\},$$

where the closure is meant in $L^2(\Gamma\backslash G)$. Then V_ϕ is a unitary representation of G.

Irreducible unitary representations V_π of G in general decompose into K-types as

$$V_\pi = \bigoplus_{\tau \in \hat{K}} m(\pi, \tau) V_\tau$$

with finite multiplicities $m(\pi, \tau)$.

A representation-theoretic point of view

Given a Hecke–Maaß cusp form ϕ on X as above, it generates a unitary representation as follows. First, we consider its right translates

$$\phi_x(g) := \phi(gx), \qquad x \in G,$$

which are not necessarily (and typically are not) functions on X but only on $\Gamma\backslash G$. Then we consider

$$V_\phi := \text{closure of the linear span of } \{\phi_x : x \in G\},$$

where the closure is meant in $L^2(\Gamma\backslash G)$. Then V_ϕ is a unitary representation of G.

Irreducible unitary representations V_π of G in general decompose into K-types as

$$V_\pi = \bigoplus_{\tau \in \hat{K}} m(\pi, \tau) V_\tau$$

with finite multiplicities $m(\pi, \tau)$. A representation is called spherical if $m(\pi, \tau_0) > 0$ for the trivial representation τ_0 of K. This is the case for V_ϕ, but not in general, e.g. classical holomorphic forms (which in fact had been discovered much before Maaß forms) generate non-spherical automorphic representations.

Hecke–Maaß cusp forms of type τ

Hecke–Maaß cusp forms of type τ

Set $Y := \Gamma\backslash G$. Let $\tau \in \hat{K}$.

Hecke–Maaß cusp forms of type τ

Set $Y := \Gamma\backslash G$. Let $\tau \in \hat{K}$.

Definition

A function $\phi : Y \to \mathbb{C}$ is a Hecke–Maaß cusp form of type τ, if it is a joint eigenfunction of $\mathcal{D}$, all the Hecke operators, and under the right-K-action, it generates a representation of K isomorphic to V_τ.

Hecke–Maaß cusp forms of type τ

Set $Y := \Gamma\backslash G$. Let $\tau \in \hat{K}$.

Definition

A function $\phi : Y \to \mathbb{C}$ is a Hecke–Maaß cusp form of type τ, if it is a joint eigenfunction of $\mathcal{D}$, all the Hecke operators, and under the right-K-action, it generates a representation of K isomorphic to V_τ.

The $\tau \in \hat{K}$ form a partially ordered set. For any V_π as above, there is a minimal $\tau(\pi)$ for which $m(\pi, \tau(\pi)) > 0$. By a theorem of Vogan, it is then necessarily 1.

Hecke–Maaß cusp forms of type τ

Set $Y := \Gamma\backslash G$. Let $\tau \in \hat{K}$.

Definition

A function $\phi : Y \to \mathbb{C}$ is a Hecke–Maaß cusp form of type τ, if it is a joint eigenfunction of $\mathcal{D}$, all the Hecke operators, and under the right-K-action, it generates a representation of K isomorphic to V_τ.

The $\tau \in \hat{K}$ form a partially ordered set. For any V_π as above, there is a minimal $\tau(\pi)$ for which $m(\pi, \tau(\pi)) > 0$. By a theorem of Vogan, it is then necessarily 1. That is, we can consider a minimal K-type in any irreducible representation, and it is a well-defined $\dim \tau$-dimensional space.

The non-spherical sup-norm problem

The non-spherical sup-norm problem

Assume that ϕ is a Hecke–Maaß cusp form of minimal type τ (i.e. the representation generated by ϕ has minimal K-type τ). What can we say about, for $\Omega \subset Y$ compact,

$$\sup_{g\in\Omega} |\phi(g)|?$$

The non-spherical sup-norm problem

Assume that ϕ is a Hecke–Maaß cusp form of minimal type τ (i.e. the representation generated by ϕ has minimal K-type τ). What can we say about, for $\Omega \subset Y$ compact,

$$\sup_{g \in \Omega} |\phi(g)|?$$

Simpler question: do not estimate ϕ alone, but consider an L^2-normalized basis $\Phi := (\phi_1, \ldots, \phi_{\dim \tau})$ of the τ-type of V_ϕ. What can we say about

$$\sup_{g \in \Omega} \underbrace{\left(|\phi_1(g)|^2 + \ldots + |\phi_{\dim \tau}(g)|^2\right)}_{=: \|\Phi(g)\|^2}?$$

The non-spherical sup-norm problem

Assume that ϕ is a Hecke–Maaß cusp form of minimal type τ (i.e. the representation generated by ϕ has minimal K-type τ). What can we say about, for $\Omega \subset Y$ compact,

$$\sup_{g \in \Omega} |\phi(g)|?$$

Simpler question: do not estimate ϕ alone, but consider an L^2-normalized basis $\Phi := (\phi_1, \ldots, \phi_{\dim \tau})$ of the τ-type of V_ϕ. What can we say about

$$\sup_{g \in \Omega} \underbrace{(|\phi_1(g)|^2 + \ldots + |\phi_{\dim \tau}(g)|^2)}_{=:\|\Phi(g)\|^2}?$$

In view of the current methodology, this is a natural simplification, it seems very hard, potentially out of reach to separate spectrally the ϕ_j's (even though we managed to do it with much pain for the seemingly much simpler $\mathrm{SL}_2(\mathbb{C})$ for ϕ's in a Wigner basis).

Methodology and a guess on the generic bound

Methodology and a guess on the generic bound

For τ, let $\chi_\tau(\cdot) := \dim \tau \cdot \mathrm{tr}(\tau(\cdot))$ be its normalized character.
Assume $f : G \to \mathbb{C}$ is a smooth bi-τ-type function
(i.e. $f = \overline{\chi_\tau} * f = f * \overline{\chi_\tau}$) which further has a compact support.
Then we may consider the right action of f on $L^2(Y)$:

$$(R(f)\psi)(g) = \int_G f(x)\psi(gx)\ dx.$$

By the assumptions on f, this makes sense, and it acts by a scalar on the V_τ component generated by ϕ.

Methodology and a guess on the generic bound

For τ, let $\chi_\tau(\cdot) := \dim \tau \cdot \mathrm{tr}(\tau(\cdot))$ be its normalized character.
Assume $f : G \to \mathbb{C}$ is a smooth bi-τ-type function
(i.e. $f = \overline{\chi_\tau} * f = f * \overline{\chi_\tau}$) which further has a compact support.
Then we may consider the right action of f on $L^2(Y)$:

$$(R(f)\psi)(g) = \int_G f(x)\psi(gx)\, dx.$$

By the assumptions on f, this makes sense, and it acts by a scalar on the V_τ component generated by ϕ.
In fact $R(f)$ acts as a scalar on all the functions of τ type which generate irreducible representations (since it is a weighted mix of right translates).

Methodology and a guess on the generic bound

For τ, let $\chi_\tau(\cdot) := \dim \tau \cdot \mathrm{tr}(\tau(\cdot))$ be its normalized character.
Assume $f : G \to \mathbb{C}$ is a smooth bi-τ-type function
(i.e. $f = \overline{\chi_\tau} * f = f * \overline{\chi_\tau}$) which further has a compact support.
Then we may consider the right action of f on $L^2(Y)$:

$$(R(f)\psi)(g) = \int_G f(x)\psi(gx)\, dx.$$

By the assumptions on f, this makes sense, and it acts by a scalar on the V_τ component generated by ϕ.
In fact $R(f)$ acts as a scalar on all the functions of τ type which generate irreducible representations (since it is a weighted mix of right translates). Assuming that $R(f)$ is a positive operator (which can be achieved by convolution squares), we may assume that the operator is positive. In ψ, we may take a smooth approximation of the identity, then by positivity we drop the other terms, and we arrive at

$$c_\Phi(f)|\Phi(g)|^2 \leq \sum_{\gamma\in\Gamma} f(g^{-1}\gamma g).$$

Methodology and a guess on the generic bound

For τ, let $\chi_\tau(\cdot) := \dim \tau \cdot \mathrm{tr}(\tau(\cdot))$ be its normalized character.
Assume $f : G \to \mathbb{C}$ is a smooth bi-τ-type function
(i.e. $f = \overline{\chi_\tau} * f = f * \overline{\chi_\tau}$) which further has a compact support.
Then we may consider the right action of f on $L^2(Y)$:

$$(R(f)\psi)(g) = \int_G f(x)\psi(gx)\ dx.$$

By the assumptions on f, this makes sense, and it acts by a scalar on the V_τ component generated by ϕ.
In fact $R(f)$ acts as a scalar on all the functions of τ type which generate irreducible representations (since it is a weighted mix of right translates). Assuming that $R(f)$ is a positive operator (which can be achieved by convolution squares), we may assume that the operator is positive. In ψ, we may take a smooth approximation of the identity, then by positivity we drop the other terms, and we arrive at

$$c_\Phi(f)|\Phi(g)|^2 \leq \sum_{\gamma\in\Gamma} f(g^{-1}\gamma g).$$

In fact, the sum on the right-hand side has $O(1)$ terms, so the guess on the generic bound depends on the relation of $c_\Phi(f)$ and f.

Non-spherical transforms

Non-spherical transforms

The relation of $c_\Phi(f)$ and f is given by the non-spherical version of Harish-Chandra's transform. In fact, to use this formula, we rather start from the $c_\Phi(f)$ side, prescribe it to, say, 1 at our favourite representation, and prescribe also a decay for other representations. By the non-spherical transform, we get an f.

Non-spherical transforms

The relation of $c_\Phi(f)$ and f is given by the non-spherical version of Harish-Chandra's transform. In fact, to use this formula, we rather start from the $c_\Phi(f)$ side, prescribe it to, say, 1 at our favourite representation, and prescribe also a decay for other representations. By the non-spherical transform, we get an f.

Question

What kind of $c_\Phi(f)$ can be prescribed?

Admittedly, we do not know the answer at the moment, but hopefully, a Paley–Wiener type (or at least a Schwartz type) function on the spectral side can be prescribed. Assume this is the case.

Non-spherical transforms

The relation of $c_\Phi(f)$ and f is given by the non-spherical version of Harish-Chandra's transform. In fact, to use this formula, we rather start from the $c_\Phi(f)$ side, prescribe it to, say, 1 at our favourite representation, and prescribe also a decay for other representations. By the non-spherical transform, we get an f.

Question

What kind of $c_\Phi(f)$ can be prescribed?

Admittedly, we do not know the answer at the moment, but hopefully, a Paley–Wiener type (or at least a Schwartz type) function on the spectral side can be prescribed. Assume this is the case.

Then by estimating the non-spherical trace function trivially for the inverse transform, and the fact that we only have $O(1)$ many terms, we get that

$$\sup_{g\in\Omega} |\Phi(g)| \ll_\Omega (\dim\tau \cdot \text{Plancherel density of } \Phi)^{1/2}.$$

This would be a clean generalization of Sarnak's bound (in fact, reproducing it for $\tau = \tau_0$, and in that case, we have the necessary Paley–Wiener theory in hand).

Winning over that: the arithmetic situation

Winning over that: the arithmetic situation

This part so far only uses that ϕ is an eigenform of $\mathcal{D}$ and we have not used the Hecke operators.

Winning over that: the arithmetic situation

This part so far only uses that ϕ is an eigenform of $\mathcal{D}$ and we have not used the Hecke operators.

If we use the Hecke operators, then the f does not have $O(1)$ many terms any more. Then the analysis of the non-spherical trace functions near K becomes relevant (which can be efficiently done!), and we arrive at a juicy matrix-counting problem. A special case is the following:

Winning over that: the arithmetic situation

This part so far only uses that ϕ is an eigenform of $\mathcal{D}$ and we have not used the Hecke operators.

If we use the Hecke operators, then the f does not have $O(1)$ many terms any more. Then the analysis of the non-spherical trace functions near K becomes relevant (which can be efficiently done!), and we arrive at a juicy matrix-counting problem. A special case is the following:

Matrix-counting problem

Assume that p is a large prime number. Prove that the number of those $n \times n$ integral matrices γ which are Smith normal form is $\mathrm{diag}(p^2, p, \dots, p, 1)$ and which satisfy $\gamma^T \gamma = p^2 I_n$ (i.e. they are orthogonal up to a scalar) is $O(p^{n-1-\eta})$ with some $\eta > 0$ fixed (might depend on n, but on p).

Winning over that: the arithmetic situation

This part so far only uses that ϕ is an eigenform of $\mathcal{D}$ and we have not used the Hecke operators.
If we use the Hecke operators, then the f does not have $O(1)$ many terms any more. Then the analysis of the non-spherical trace functions near K becomes relevant (which can be efficiently done!), and we arrive at a juicy matrix-counting problem. A special case is the following:

Matrix-counting problem

Assume that p is a large prime number. Prove that the number of those $n \times n$ integral matrices γ which are Smith normal form is $\mathrm{diag}(p^2, p, \ldots, p, 1)$ and which satisfy $\gamma^T \gamma = p^2 I_n$ (i.e. they are orthogonal up to a scalar) is $O(p^{n-1-\eta})$ with some $\eta > 0$ fixed (might depend on n, but on p).

We know how to treat this.

Winning over that: the arithmetic situation

This part so far only uses that ϕ is an eigenform of $\mathcal{D}$ and we have not used the Hecke operators.

If we use the Hecke operators, then the f does not have $O(1)$ many terms any more. Then the analysis of the non-spherical trace functions near K becomes relevant (which can be efficiently done!), and we arrive at a juicy matrix-counting problem. A special case is the following:

Matrix-counting problem

Assume that p is a large prime number. Prove that the number of those $n \times n$ integral matrices γ which are Smith normal form is $\operatorname{diag}(p^2, p, \ldots, p, 1)$ and which satisfy $\gamma^T \gamma = p^2 I_n$ (i.e. they are orthogonal up to a scalar) is $O(p^{n-1-\eta})$ with some $\eta > 0$ fixed (might depend on n, but on p).

We know how to treat this. It gives

$$\sup_{g \in \Omega} |\Phi(g)| \ll_\Omega (\dim \tau \cdot \text{Plancherel density of } \Phi)^{1/2-\delta}$$

with some $\delta > 0$ depending only on n.

Erratum

Only essential corrections, typos, etc. are not taken into consideration.

- on slide 18, the function f is also assumed to be K-central, i.e. $f(k^{-1}xk) = f(x)$ for any $x \in G$ and $k \in K$