New advances in the study of the ternary purely exponential Diophantine equation $a^x + b^y = c^z$

Maohua Le, Reese Scott, Robert Styer

Lingnan Normal University, Independent, Villanova University

12 May 2023

Let $N(a,b,c)$ be the number of solutions in positive integers (x,y,z) to the equation

$$a^x + b^y = c^z, a, b, c \in \mathbb{Z}^+, b > a > 1, \gcd(a,b) = 1, \qquad (1)$$

with a, b, c not perfect powers.

Conjecture (Cases with $N(a, b, c) > 1$)

$$
\begin{aligned}
N(3,5,2) &= 3 : (x,y,z) = (1,1,3),(3,1,5),(1,3,7),\\
N(3,13,2) &= 2 : (x,y,z) = (1,1,4),(5,1,8),\\
N(3,10,13) &= 2 : (x,y,z) = (1,1,1),(7,1,3),\\
N(2,89,91) &= 2 : (x,y,z) = (1,1,1),(13,1,2),\\
N(2,3,11) &= 2 : (x,y,z) = (1,2,1),(3,1,1),\\
N(2,3,35) &= 2 : (x,y,z) = (5,1,1),(3,3,1),\\
N(2,3,259) &= 2 : (x,y,z) = (8,1,1),(4,5,1),\\
N(2,5,133) &= 2 : (x,y,z) = (7,1,1),(3,3,1),\\
N(2,91,8283) &= 2 : (x,y,z) = (13,1,1),(1,2,1),\\
N(3,13,2200) &= 2 : (x,y,z) = (7,1,1),(1,3,1),\\
N(2,5,3) &= 2 : (x,y,z) = (1,2,3),(2,1,2),\\
N(2,3,5) &= 2 : (x,y,z) = (1,1,1),(4,2,2),\\
N(2,7,3) &= 2 : (x,y,z) = (1,1,2),(5,2,4),\\
N(2,2^r-1,2^r+1) &= 2 : (x,y,z) = (1,1,1),(r+2,2,2),\ r>3.
\end{aligned}
$$

The Pillai Case

A special case of the title equation is the familiar Pillai equation: let $P(d,b,c)$ equal the number of solutions (x,y,z) (with c and b not perfect powers) to

$$c^z - b^y = d, d > 0, \gcd(b,c) = 1. \tag{2}$$

Bennett conjectured that $P(d,b,c) = 1$ except for the following cases:

$$\begin{aligned}
P(1,2,3) &= 2 : (y,z) = (1,1),(3,2),\\
P(3,5,2) &= 2 : (y,z) = (1,3),(3,7),\\
P(5,3,2) &= 2 : (y,z) = (1,3),(3,5),\\
P(13,3,2) &= 2 : (y,z) = (1,4),(5,8),\\
P(10,3,13) &= 2 : (y,z) = (1,1),(7,3),\\
P(89,2,91) &= 2 : (y,z) = (1,1),(13,2).
\end{aligned}$$

Conjecture (Revised by removing trivial rearrangements)

Let $N(a,b,c)$ equal the number of solutions (x,y,z) to $a^x + b^y = c^z$, now allowing $a = 1$ (in which case $N(a,b,c)$ is the number of solutions (y,z) to $a^x + b^y = c^z$). Then $N(a,b,c) = 1$ except for the following cases and trivial rearrangements of these cases:

$$
\begin{aligned}
N(1,2,3) &= 2 : (y,z) = (1,1),(3,2),\\
N(3,5,2) &= 3 : (x,y,z) = (1,1,3),(3,1,5),(1,3,7),\\
N(3,13,2) &= 2 : (x,y,z) = (1,1,4),(5,1,8),\\
N(3,10,13) &= 2 : (x,y,z) = (1,1,1),(7,1,3),\\
N(2,89,91) &= 2 : (x,y,z) = (1,1,1),(13,1,2),\\
N(2,5,3) &= 2 : (x,y,z) = (1,2,3),(2,1,2),\\
N(2,3,5) &= 2 : (x,y,z) = (1,1,1),(4,2,2),\\
N(2,7,3) &= 2 : (x,y,z) = (1,1,2),(5,2,4),\\
N(2,2^r-1,2^r+1) &= 2 : (x,y,z) = (1,1,1),(r+2,2,2),\ r > 3.
\end{aligned}
$$

Double Solutions

Known cases of (a, b, c) giving exactly two solutions to $a^x + b^y = c^z$ are of three types:

- Cases in which at least one exponent is the same in both solutions, giving a Pillai case.
- (a, b, c) or (a, b, c^2) equals $(2, 2^r - 1, 2^r + 1)$, $r > 0$, giving a Mersenne-Fermat case.
- $(a, b, c) = (2, 5, 3)$: $(x, y, z) = (1, 2, 3)$, $(2, 1, 2)$, the only case not related to a Pillai case or the Mersenne-Fermat infinite family.

Eight Cases we will consider

- *atmost2* versus *atmost1*
- $c^z - b^y = d$ (Pillai) versus $a^x + b^y = c^z$ (general)
- Prime bases versus unrestricted positive integer bases

Methods Used

Difference between methods needed for prime bases and methods needed for unrestricted bases.

RED indicates prime bases.

BLUE indicates unrestricted bases.

PURPLE indicates method of proof uses ideals in imaginary quadratic fields or other elementary methods.

GREEN indicates method of proof uses lower bounds on linear forms in logs or other deeper methods.

$\checkmark$ indicates complete proof for the case in question.

atmost2Pillai ($c^z - b^y = d$)

Bennett, On some exponential equations of S. S. Pillai, CJM, 2001.

c prime: at most two solutions to $c^z - b^y = d$ ✓
(Section 2 of Bennett 2001 summarizes earlier results for c prime which include *atmost2*).
Ideals in imaginary quadratic fields is all that is needed (elementary)

c any positive integer: *atmost2* solutions to $c^z - b^y = d$ ✓
(Section 3 of Bennett 2001 gives the proof.)
Lower bounds on linear forms in logs used in proof (Mignotte 1998).

atmost2 for $a^x + b^y = c^z$

Early results with prime bases: Nagell (1958), Le (1985), Cao (1991).
c prime: at most two solutions to $a^x + b^y = c^z$ ✓ Shown using ideals in imaginary quadratic fields (Theorem 6 of Scott, 1993)

At most two solutions to $a^x + b^y = c^z$ (allowing composite bases) with one exceptional (a, b, c) (Miyazaki and Pink, Number of solutions to a special type of unit equation in two variables, to appear in Amer. J. Math. ✓
(Details follow.)

atmost2 for general case ($a^x + b^y = c^z$)

Best available published bound before recent work:
$N(a, b, c) < 2^{36}$ (derived from Beukers and Schlickewei, 1996).
Possible unpublished bound: $N(a, b, c) < 200$ (Hirata-Kohno?)
Recent results showing $N(a, b, c) \leq 2$:

- c odd (Scott and Styer. Number of Solutions to $a^x + b^y = c^z$, Debrecen, 2016.)
- c even and $\max(a, b, c) > 10^{62}$ (Hu and Le. An upper bound for the number of solutions of ternary purely exponential Diophantine equations, II, Debrecen 2019.)
- c even and $\max(a, b, c) \leq 10^{62}$ (Miyazaki and Pink. Number of solutions to a special type of unit equation in two variables, to appear in Amer. J. Math.) ✓

Miyazaki and Pink ✓

Two main steps:

- Results of Hu and Le (2009) used and improved to reduce bound on $\max(a, b, c)$ when 3 solutions exist.
- Assuming 3 solutions, (x_i, y_i, z_i), $i = 1, 2, 3$, with $z_1 \leq z_2 \leq z_3$, sharp bounds on x_1, y_1, z_1, x_2, y_2, z_2 are obtained using 2-adic arguments made possible by assuming c is even (using Scott and Styer 2016).
- Additional theoretical maneuvers.
- Extensive calculations.

c even and $\max\{a, b, c\} > 10^{62}$ (Hu and Le, 2019)

Earlier result of Hu and Le:

$$\max\{x, y, z\} < 6500(\log(\max\{a, b, c\}))^3$$

linear forms in logs (Laurent, Mignotte, Nesterenko, 1995)
linear forms in p-adic logs (Bugeaud, 1999)

Now elementary approaches suffice, using c even:

Lemma 4.6 bounds $\max\{a, b, c\}$ when 3 solutions satisfying certain conditions exist.

When Lemma 4.6 does not apply, continued fractions are derived from the exponents x_i, y_i, z_i which lead to an upper bound on $\max\{a, b, c\}$.

c odd (Scott and Styer 2018)

Lower bounds on linear forms in logs not needed.
Consider solutions (A, B, z) to the equation

$$A + B = c^z \qquad (*)$$

where $c > 1$ and $AB = \prod_{i=1}^{n} p_i^{\alpha_i}$, $\alpha_i > 0$.

$$[A - B \pm 2\sqrt{-AB}] = \mathfrak{c}^{2z}.$$

Let ω be the number of primes dividing c: number of solutions to (*) bounded by $2^{n+\omega-1}$. To improve this to $2^{n-1}+1$:
Let p be the number of parity classes possible for α_i, let q be the number of ideal factorizations possible for a given parity class of α_i.

$$pq = 2^{n-1}.$$

(Scott and Styer, Two terms with known prime divisors adding to a power, Debrecen, 2018.)

atmost1Pillai ($c^z - b^y = d$)

c prime, (Bennett 2001): b and d not necessarily prime.
At most one solution when $c = 2$, 3, 5, 17, 257, 65537.

$c = 2$ handled in Section 2 of Bennett 2001.

c a Fermat prime handled in Section 7 of Bennett 2001.

b, c prime, $b \not\equiv 1 \bmod 12$ (Scott Styer 2004):

- if $c^z - b^y = d$ has two solutions, c must be a base b Wieferich prime, with five listed exceptions (linear forms in logs).
- At most one solution to $c^z - b^y = d$ when either $b > \frac{d}{16}$ or $c > \frac{d}{31}$, excepting listed (b, c, d) (linear forms in logs).

Bennett 2001:

- $d \geq b^{2c^2 \log(c)}$
- at most one solution with $b^y \geq 6000d$.

Using lower bounds on linear forms in logs.

atmost1 ($a^x + b^y = c^z$)

a, b, c primes:
New results for prime bases discussed in the remaining slides.
"On a conjecture concerning the number of solutions to $a^x + b^y = c^z$, II." Le, Scott, Styer, arXiv:2211.13378

Allowing composite a, b, c
Miyazaki and Pink (arXiv:2205.11217)
At most one solution (with listed exceptions) for $c = 2, 3, 5, 6, 17, 257, 65537$.
Note that $c = 6$ was not even handled for Pillai case!
$c = 6$ is the first composite value completely handled.
Infinite number of values of c reduced to a finite (albeit impractical) search.
This case (allowing composite a, b, c) already handled in the Debrecen seminar last November by Miyazaki.

atmost1 for general case ($a^x + b^y = c^z$) for prime bases

Quite different methods for prime bases than for composite bases. Let $S(a, b, c)$ be the number of solutions in positive integers (x, y, z) to the equation

$$a^x + b^y = c^z, a, b, c \text{ prime }, a < b.$$

Conjecture

For a, b, and c distinct primes with $a < b$, we have $S(a, b, c) \leq 1$, except for
(i) $S(2, 3, 5) = 2$, $(x, y, z) = (1, 1, 1)$ *and* $(4, 2, 2)$.
(ii) $S(2, 3, 11) = 2$, $(x, y, z) = (1, 2, 1)$ *and* $(3, 1, 1)$.
(iii) $S(2, 5, 3) = 2$, $(x, y, z) = (1, 2, 3)$ *and* $(2, 1, 2)$.
(iv) $S(2, 7, 3) = 2$, $(x, y, z) = (1, 1, 2)$ *and* $(5, 2, 4)$.
(v) $S(3, 5, 2) = 3$, $(x, y, z) = (1, 1, 3)$, $(1, 3, 7)$, *and* $(3, 1, 5)$.
(vi) $S(3, 13, 2) = 2$, $(x, y, z) = (1, 1, 4)$ *and* $(5, 1, 8)$.

atmost1 for general case for prime bases, continued

Well known elementary results summarized in Section 2 of Bennett 2001 immediately give:
If two solutions (x_1, y_1, z_1) and (x_2, y_2, z_2) to $a^x + b^y = c^z$ occur and (a, b, c) is not equal to $(2, 3, 5)$, $(2, 3, 11)$, $(2, 5, 3)$, $(2, 7, 3)$, $(3, 5, 2)$, $(3, 13, 2)$, we must have

$$2^{x_1} + b^{y_1} = c$$

and

$$2^{x_2} + b^{y_2} = c^{z_2}, z_2 > 1.$$

Red exponents are even, blue exponents are odd.

atmost1 for general case for prime bases, continued

From these two equations it follows that we must have one of six cases:

$b \equiv 2 \bmod 3$:

$x_2 > 1,\ y_2 > 1,$

$x_2 > 1,\ y_2 = 1,$

$x_2 = 1,\ y_2 > 1.$

$b \equiv 1 \bmod 3$:

$b \equiv 13 \bmod 24,\ c \equiv 5 \bmod 24,$

$b \equiv 13 \bmod 24,\ c \equiv 17 \bmod 24,$

$b \equiv 1 \bmod 24,\ c \equiv 17 \bmod 24.$

These six cases are handled in six completely different ways.

atmost1 for general case for prime bases, continued

$b \equiv 2 \bmod 3$, $x_2 > 1$, $y_2 > 1$: handled using deep result of Bennett and Skinner (2004).
$b \equiv 2 \bmod 3$, $x_2 > 1$, $y_2 = 1$: handled using Bauer and Bennett (2002).
$b \equiv 2 \bmod 3$, $x_2 = 1$, $y_2 > 1$: handled using Bennett (2008).
Deep methods required.

atmost1 for general case for prime bases, continued

$b \equiv 13 \bmod 24$, $c \equiv 5 \bmod 24$: handled using special properties of special continued fractions.
$b \equiv 13 \bmod 24$, $c \equiv 17 \bmod 24$: handled using the theory of quartic residues (as proved by Dirichlet).
$b \equiv 1 \bmod 24$, $c \equiv 17 \bmod 24$: not yet completely handled.
Methods used here are elementary.

The unhandled case, prime bases

The case $b \equiv 1 \bmod 24$, $c \equiv 17 \bmod 24$ gives more parity restrictions on the exponents:
If two solutions (x_1, y_1, z_1) and (x_2, y_2, z_2) to $a^x + b^y = c^z$ occur and (a, b, c) is not equal to $(2, 3, 5)$, $(2, 3, 11)$, $(2, 5, 3)$, $(2, 7, 3)$, $(3, 5, 2)$, $(3, 13, 2)$, we must have

$$2^{x_1} + b^{y_1} = c$$

and

$$2^{x_2} + b^{y_2} = c^{z_2}, z_2 > 1.$$

Red exponents are even, blue exponents are odd.
From these two equations the following restrictions are derived:

The unhandled case, continued

- $a = 2$, $b \equiv 1 \bmod 48$, $c \equiv 17 \bmod 48$;
- $b > 10^9$, $c > 10^{18}$;
- at least one of the multiplicative orders $u_b(c)$ or $u_c(b)$ must be odd (where $u_p(n)$ is the least integer t such that $n^t \equiv 1 \bmod p$);
- 2 must be an octic residue modulo c except for one specific case;
- $2 \mid v_2(b-1) \leq v_2(c-1)$ (where $v_2(n)$ satisfies $2^{v_2(n)} \parallel n$);
- there must be exactly two solutions (x_1, y_1, z_1) and (x_2, y_2, z_2) with $1 = z_1 < z_2$ and either $x_1 \geq 28$ or $x_2 \geq 88$.

Unlikelihood of further solutions

$$Q = \frac{\log(c)}{\log(rad(abc))}$$

Then for the equation $2^{x_2} + b^{y_2} = c^{z_2}$ we have

$$Q = \frac{z_2 \log(c)}{\log(2) + \log(b) + \log(c)} \geq \frac{3 \log(c)}{(3/2)\log(c) + \log(2)}$$
$$= 2 - \frac{2\log(2)}{(3/2)\log(c) + \log(2)} > 1.97.$$

The highest value for Q found in recent researches on the *abc* conjecture is $Q = 1.62991$ for $(a, b, c) = (2, 3^{10} \cdot 109, 23^5)$. If $z_2 > 3$, then we have $Q > 3.29$: if a conjecture of Tenenbaum (quoted in Section B19 of Guy) is true, then $Q = 3.29$ is impossible, so that $z_2 = 3$.

bounds on b and c for a, b, c all prime

If $a^x + b^y = c^z$ has more than one solution:

$$b > 10^9$$

$$c > 10^{18}$$

Some key ideas: $z_1 = 1$, and z_2 must divide the class number of $\mathbb{Q}(\sqrt{-b})$. Examine exponents modulo small primes to eliminate values of $b < 10^9$.
Details in Section 5 of "On a conjecture concerning the number of solutions to $a^x + b^y = c^z$ ", Le and Styer, BAMS, 2022.

Bounds on a, b, c not necessarily prime

Styer recently showed that the general conjecture (allowing composite bases) holds for $a, b < 1000$, $c < 10^{10}$.

- The c even case uses the ideas of Miyazaki and Pink (to appear).
- The c odd case also includes ideas of Scott (1993).

The unhandled case, with prime bases

$b \equiv 1 \bmod 48$, $c \equiv 17 \bmod 48$:
If two solutions (x_1, y_1, z_1) and (x_2, y_2, z_2) to $a^x + b^y = c^z$ occur and (a, b, c) is not equal to $(2, 3, 5)$, $(2, 3, 11)$, $(2, 5, 3)$, $(2, 7, 3)$, $(3, 5, 2)$, $(3, 13, 2)$, we must have

$$2^{x_1} + b^{y_1} = c$$

and

$$2^{x_2} + b^{y_2} = c^{z_2}, z_2 > 1.$$

Red exponents are even, blue exponents are odd.

These two equations lead to many restrictions as outlined above. Will new methods be required to finish this case?