Multiply gleeful numbers

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- For positive integers n and k let $f_k(n)$ be the number of representations of n as a sum of kth powers of consecutive primes.

- Moser **1963** puts

$$s_k(x) = \sum_{n \le x} f_k(n)$$

and shows that

$$s_1(x) \sim x \log 2 \qquad \text{as} \qquad x \to \infty.$$

Upper and lower bounds for $s_2(x)$ and $s_k(x)$ (for any $k > 1$) appear in some recent works of Moore, Sorenson **2025**.

- That preprint presents various heuristics concerning positive integers n such that $f_k(n) f_{k'}(n) > 0$ for some $k' > k \ge 2$.

Definition

A positive integer n with $f_k(n) > 0$ is called *gleeful* (of k-gleeful).

Numbers $n \leq x$ with $f_k(n) > 0$

• Let

$$\mathcal{G}_k(x) = \{n \leq x : f_k(n) > 0\}.$$

• O'Sullivan, Sorenson, Stahl **2024** showed that for all $k \geq 2$, one has

$$c_k \frac{x^{2/(k+1)}}{(\log x)^{2k/(k+1)}} < \#\mathcal{G}_k(x) \leq d_k \frac{x^{2/(k+1)}}{(\log x)^{2k/(k+1)}}, \quad x > x_k,$$

with some positive constants $c_k,\ d_k$ which are explicit.

• In fact, assume

$$n = p_i^k + p_{i+1}^k + \cdots + p_{i+\ell-1}^k.$$

If $n \leq x$, we have

$$p_i^k \ell \leq x \quad \text{therefore} \quad p = p_i \leq \sqrt[k]{\frac{x}{\ell}}.$$

• Letting L be the largest ℓ can be and summing up the above, we get

$$\#\mathcal{G}_k(x) \ll \sum_{\ell\le L} \pi\left(\sqrt[k]{\frac{x}{\ell}}\right) \ll \frac{x^{1/k}}{\log x}\sum_{\ell\le L}\frac{1}{\sqrt[k]{\ell}} \ll \frac{x^{1/k}L^{1-1/k}}{\log x}.$$

• How large can L be? Well, for sure

$$p_1^k+\cdots+p_L^k\le x.$$

• The left–hand side is

$$\gg_k \frac{p_L^{k+1}}{(k+1)\log p_L} \gg_k L^{k+1}(\log L)^k.$$

• Thus,

$$L^{k+1}(\log L)^k \ll_k x, \quad \text{therefore} \quad L \ll \frac{x^{1/(k+1)}}{(\log x)^{k/(k+1)}}.$$

• Putting the above together we get

$$\#\mathcal{G}_k(x) \ll \frac{x^{1/k} x^{(1-1/k)/(k+1)}}{(\log x)^{1+k/(k+1)(1-1/k)}} = \frac{x^{2/(k+1)}}{(\log x)^{2k/(k-1)}}.$$

• We can interpret this by saying that the probability that n is k-gleeful is

$$\frac{1}{n^{\alpha_k + o(1)}} \quad \text{as} \quad n \to \infty,$$

where

$$\alpha_k = 1 - \frac{2}{k+1}.$$

• Based on this Moore, Sorenson **2025** make a number of conjectures on n with $f_k(n) > 0$ and $f_{k'}(n) > 0$.

Conjecture

For $k < k'$ and $k \ge 3$ or $k' \ge 5$, there are only finitely many n with $f_k(n) > 0,\ f_{k'}(n) > 0$.

This is based on the fact that

$$\alpha_k + \alpha_{k'} = 2 - \frac{2}{k+1} - \frac{2}{k'+1} > 1$$

in such instances.

Conjecture

For $k' = 3, 4$ there are infinitely many n with $f_2(n) > 0$ and $f_{k'}(n) > 0$.

• They found

$23939 = 17^3 + 19^3 + 23^3 = 23^2 + 29^2 + 31^2 + 37^2 + 41^2 + 43^2 + 47^2 + 53^2 + 59^2 + 61^2 + 67^2$.

Conditional proof that $f_2(n)f_4(n) > 0$ infinitely often

Our result relies on the following conjecture known as Schinzel's Hypothesis H.

Conjecture: Schinzel's Hypothesis H

Let $k \geq 2$ and $f_i(X) \in \mathbb{Z}[X]$ be irreducible polynomials with positive leading terms for $i = 1, \ldots, k$. Assume that for all primes p there exists n such that

$$p \nmid f_1(n)f_2(n) \cdots f_k(n).$$

Then there exist infinitely many positive integers n such that $f_1(n), \ldots, f_k(n)$ are all primes.

Our result is the following.

Theorem

Assume Schinzel's Hypothesis H. Then there are infinitely many positive integers n such that $f_2(n)f_4(n) > 0$.

The construction

- We aim to construct positive integers m such that

$$m = \sum_{j=1}^{25} p_j^2 = \sum_{j=1}^{25} q_j^4,$$

where $p_1 < \cdots < p_{25}$ and $q_1 < \cdots < q_{25}$ are consecutive primes.

- We start with the degree 4 side. We choose linear forms

$$X, \ X - a_i, \ X + a_i \quad \text{in} \quad \mathbb{Z}[X] \quad \text{for} \quad i = 1, \ldots, 12,$$

which later we aim to specialize in some input n so that to obtain primes. Then

$$\begin{aligned} M(X) \quad &:= \quad X^4 + \sum_{i=1}^{12}((X - a_i)^4 + (X + a_i)^4) \\ &= \quad 25X^4 + 12(\sum_{i=1}^{12} a_i^2)X^2 + 2\sum_{i=1}^{12} a_i^4. \end{aligned}$$

- We want to complete a square in the above so we calculate

$$\delta := \frac{12(\sum_{i=1}^{12} a_i^2)}{10} = \frac{6}{5}\left(\sum_{i=1}^{12} a_i^2\right),$$

and get

$$M(X) = (5X^2+\delta)^2+\nu, \quad \text{where} \quad \nu := 2\sum_{i=1}^{12} a_i^4 - \left(\frac{6}{5}\left(\sum_{i=1}^{12} a_i^2\right)\right)^2.$$

- We now look at the quadratic side. We take linear forms

$$Y, \ Y - b_j, \ Y + b_j \quad \text{in} \quad \mathbb{Z}[Y] \quad \text{for} \quad j = 1, \ldots, 12,$$

and we calculate

$$N(Y) = Y^2 + \sum_{j=1}^{12} (Y - b_j)^2 + (Y + b_j)^2 = 25Y^2 + 2\sum_{j=1}^{12} b_j^2.$$

We want $M(X) = N(Y)$, which gives

$$5Y = 5X^2 + \delta, \qquad 2\sum b_j^2 = \nu.$$

• Numerically, we choose

$$a_i = 2i - 1, \qquad i = 1, \ldots, 11, \quad \text{and} \qquad a_{12} = 123.$$

Then

$$\delta = 20280, \quad \nu = 47518520.$$

We then have $\nu/2 = 23759260$ which can be written as a sum of 12 distinct squares, for example, as

$$1700^2 + 1701^2 + 1702^2 + 1703^2 + 1704^2 + 1705^2 + 1706^2 + 1707^2 + 100^2 + 160^2 + 492^2 + 516^2.$$

• So, we take

$b_j = 1699 + j,\ 1 \le j \le 8,\ b_9 = 100,\ b_{10} = 160,\ b_{11} = 492,\ b_{12} = 516.$

With this choice,

$$Y = X^2 + \frac{\delta}{5} = X^2 + 4056.$$

Note that if replace a_j by $a_j\Lambda$ and b_j by $b_j\Lambda^2$ for any integer Λ then the same formulas apply. Thus, we take

$$\begin{aligned} p_j(X) &:= Y - b_j = X^2 + (\delta/5 - b_j)\Lambda^2, \quad j = 1, \ldots, 12, \\ p_{13}(X) &:= Y = X^2 + (\delta/5)\Lambda^2, \\ p_j(X) &:= Y + b_{j-13} = X^2 + (\delta/5 + b_{j-13})\Lambda^2, \qquad j = 14, \ldots, 25. \end{aligned}$$

We take

$$\begin{aligned} q_j(X) &:= X - a_j\Lambda \qquad j = 1, \ldots, 12, \\ q_{13}(X) &:= X, \\ q_j(X) &:= X + a_{j-13}\Lambda. \qquad j = 14, \ldots, 25. \end{aligned}$$

• We take

$\Lambda = 2{\cdot}3{\cdot}5{\cdot}7{\cdot}11{\cdot}13{\cdot}17{\cdot}19{\cdot}23{\cdot}29{\cdot}31{\cdot}37{\cdot}41{\cdot}43{\cdot}47{\cdot}53{\cdot}59{\cdot}61{\cdot}67{\cdot}71{\cdot}73$

to be the product of all the primes $p \le 73$. Numerically,

$$\Lambda = 40729680599249024150621323470.$$

Let

$$P(X) := \prod_{j=1}^{25} p_j(X)q_j(X).$$

This is a polynomial of degree 75. If $p \le 75$, then

$$P(X) \equiv X^{75} \pmod{p},$$

because $p \mid \Lambda$.

• This shows that there exists $x_0 \in \{0, \ldots, p-1\}$ such that $P(x_0) \not\equiv 0 \pmod{p}$ for all $p \le 75$ (for example, we can take $x_0 = 1$ for all such primes).

• The same is true for $p > 75$ because for such primes the polynomial $P(X)$ has at most $\deg(P) = 75 < p$ roots modulo p.

• Since each $p_j(X)$ and $q_j(X)$ for $j = 1, \ldots, 25$ is monic and irreducible, it follows, by Schinzel's Hypothesis H that there are infinitely many n such that $p_j(n)$ and $q_j(n)$ are primes for all $j = 1, \ldots, 25$.

• As we remarked,

$$\sum_{j=1}^{25} p_j(n)^2 = \sum_{j=1}^{25} q_j(n)^4.$$

So, the number m which is the common value of the left and right hand side of the above expression has $f_2(m) > 0$ and $f_4(m) > 0$, provided $p_j(n)$ are consecutive primes and $q_j(n)$ are consecutive primes for such n.

• Well, let us insure that we can find such n. We put $b_{13} := 0$ and let

$$I := [(\delta/5-1707)\Lambda^2, (\delta/5+1707)\Lambda^2] \backslash \{(\delta/5 \pm b_j)\Lambda^2 : 1 \le j \le 13\}.$$

• Let $K := \# I$ and

$$I = \{u_1, u_2, \ldots, u_K\},$$

where the above are the elements of I labelled increasingly. Select a finite set of K primes inductively $\{r_k\}_{1 \le k \le K}$ all larger than $10^5\Lambda^2$ such that

(i) $r_{k+1} > r_k$;

(ii) The polynomial $X^2 + u_k$ has an integer root x_k modulo r_k.

• Put also $a_{13} = 0$ and let

$$J := [-123\Lambda, 123\Lambda] \backslash \{\pm a_j \Lambda : 1 \le j \le 13\} = \{v_1, \ldots, v_L\},$$

where $L := \#J$ and $v_1, \ldots, v_L$ are the elements in J labelled increasingly.

• Let $\{s_\ell\}_{1 \le \ell \le L}$ be primes larger than $10^5\Lambda^2$ which are distinct from $\{r_1, \ldots, r_K\}$. Let

$$M := \prod_{k=1}^{K} r_k \prod_{\ell=1}^{L} s_\ell,$$

and let x_0 be such that

$$\begin{array}{lllll} x_0 & \equiv & x_k & \pmod{r_k} & \text{for} \quad 1 \le k \le K, \\ x_0 & \equiv & -v_\ell & \pmod{s_\ell} & \text{for} \quad 1 \le \ell \le L. \end{array}$$

The fact that the above system is solvable follows from the way we have chosen the primes r_j, s_ℓ for $1 \le j \le K$, $1 \le \ell \le L$ and from the Chinese Remainder Theorem.

• Now we apply Schinzel's Hypothesis H to the polynomials

$$p_j(MX + x_0) \quad \text{and} \qquad q_j(MX + x_0) \quad \text{for} \quad j = 1, \ldots, 25$$

as polynomials in the variable X. We note that the condition that for every prime number p there is n_0 such that $Q(n_0) \not\equiv 0 \pmod{p}$ where

$$Q(X) := P(MX + x_0)$$

is still satisfied.

• Indeed, this is trivially satisfied if $p \nmid M$, while if $p \mid M$ then we can take $n_0 = 0$ since $P(x_0)$ is not zero modulo p. Indeed, $P(x_0)$ being a multiple of p means that p divides one of

$$x_0^2 + (\delta/5 \pm b_j)\Lambda^2, \qquad \text{or} \qquad x_0 \pm a_j\Lambda,$$

for $j = 1, \ldots, 13$. However, p also divides one of

$$x_0^2 + u_k \qquad 1 \leq k \leq K, \qquad \text{or} \qquad x_0 + v_\ell, \qquad 1 \leq \ell \leq L.$$

• Thus, p divides one of

$$(\delta/5 \pm b_j)\Lambda^2 - u_k, \qquad (\delta/5 \pm b_j)\Lambda^2 + v_\ell^2, \quad \pm a_j\Lambda + v_\ell, \qquad a_j^2\Lambda^2 + u_k \tag{1}$$

for $1 \le j \le 13, 1 \le k \le K, 1 \le \ell \le L$.

• The first and third expressions are nonzero by constructions since the u_k's the v_ℓ's are the elements in their intervals

$$[(\delta/5 - 1707)\Lambda^2, (\delta/5 + 1707)\Lambda^2] \quad \text{and} \qquad [-123\Lambda, 123\Lambda]$$

which are not of the form

$$(\delta/5 \pm b_j)\Lambda^2 \quad \text{or} \quad \pm a_j\Lambda \quad \text{for} \quad 1 \le j \le 13,$$

respectively.

• The second and the fourth ones are also nonzero since they are positive.

- Now the contradiction comes from the fact that the sizes of the above nonzero numbers are

$$< 10^5\Lambda^2,$$

so they cannot be divisible by the prime

$$p \in \{r_1, \ldots, r_K\} \cup \{s_1, \ldots, s_L\}.$$

- Thus, there are infinitely many n such that

$$(Mn + x_0)^2 + (\delta/5 - b_j)\Lambda^2, \quad (Mn + x_0) \pm a_j\Lambda, \quad j = 1, \ldots, 13$$

are all primes. If the first 25 or the last 25 are not consecutive primes, it follows that there exists either $1 \le k \le K$ or $1 \le \ell \le L$ such that

$$(Mn + x_0)^2 + u_k, \quad \text{or} \quad Mn + x_0 + s_\ell$$

is prime, but this is impossible since the above numbers are divisible by r_k and s_ℓ, respectively, by our construction.

- This finishes the argument.

Comment

The Bateman-Horn conjecture is a qualitative form of the Schinzel Hypothesis H and predicts for a large positive real number T an asymptotic for the number of $n \leq T$ such that

$$f_1(n), \cdots, f_k(n)$$

are primes in case

$$f_1(X), \ldots, f_k(X)$$

satisfy the hypothesis of Schinzel's Hypothesis H. For our construction, it predicts that the number of $m \leq T$ such that

$$f_2(m) f_4(m) > 0$$

is

$$\gg \frac{T^{1/4}}{(\log T)^{50}}.$$

- We give no further details.

$f_2(n)$ is unbounded

We use a similar method to prove the following result.

Theorem

Assume Schinzel's Hypothesis H. Then

$$\limsup_{n\to\infty} f_2(n) = \infty.$$

The construction

- Let $u \geq 1$ be an integer.
- Put $k := k_u = 2^{2u+1}$ and consider the quadratic polynomial

$$Q_u(X) := \sum_{i=1}^{k_u} \left((X - a_{i,u})^2 + (X + a_{i,u})^2 \right) = 2k_u X^2 + 2 \sum_{i=1}^{k_u} a_{i,u}^2.$$

- Here, we assume that

$$a_{1,u} > a_{2,u} > \cdots > a_{k_u,u} > 0$$

are integers.

- We aim to choose positive integers m_u such that

$$Q_u(m_u) = Q_v(m_v) \quad \text{for} \quad u, v = 1, 2, \ldots, T. \tag{2}$$

• If we denote by N the common value of the numbers shown at (2) and if furthermore:

(i) $m_u - a_{1,u},\ m_u - a_{2,u}, \ldots, m_u - a_{k_u,u},\ m_u + a_{k_u,u}, \ldots, m_u + a_{1,u}$ are primes for all $u = 1, \ldots, T$;

(ii) the primes mentioned at (i) above are consecutive primes in the sequence of prime numbers;

then obviously

$$f_2(N) \geq T.$$

• Since T is arbitrary, we get the desired result.

• It remains to justify that we can meet conditions (i) and (ii) above.

• In order for (2) to hold it suffices that

$$2k_u m_u^2 = 2k_v m_v^2 \quad \text{and} \quad 2\sum_{i=1}^{k_u} a_{i,u}^2 = 2\sum_{i=1}^{k_v} a_{i,v}^2.$$

• The first condition above implies $2^{u+1}m_u = 2^{v+1}m_v$ for $u, v = 1, \ldots, T$. This is fulfilled if we choose $m_T = n$ and $m_i = 2^{T-i}n$ for $i = 1, \ldots, T-1$. For the second part, we look for an integer M such that

$$M = \sum_{i=1}^{k_u} a_{i,u}^2, \qquad u = 1, \ldots, T. \tag{3}$$

• We will want some other things from the integers $a_{i,j}$ for $j = 1, \ldots, T$ and $i = 1, \ldots, k_j$. Let

$$W = 2\sum_{u=1}^{T} k_u = \frac{2^{2T+4} - 16}{3}.$$

• This gives the total number of linear forms

$$m_u \pm a_{i,u} = 2^{T-u}n \pm a_{i,u}, \quad \text{for} \quad i = 1, \ldots, k_u \; u = 1, \ldots, T$$

in the variable n appearing in (i).

- We would like to argue that there are numbers M which have T representations as in (3), where all $a_{i,u}$ for $1 \le i \le k_u$ and $u = 1, \ldots, T$ have the property that they are not only distinct but also coprime to all primes $q \le W$.
- We choose M large, $M \equiv 8 \pmod{24}$.
- For each $u = 1, \ldots, T$, we choose

$$a_{k_u,u} < a_{k_u-1,u} < \ldots < a_{5,u} \le M^{1/3}$$

to be such that they are coprime to

$$P := \prod_{q \le W} q.$$

Note that

$$\sum_{j=5}^{k_u} a_{j,k_u}^2 \equiv k_u - 4 \pmod{24} \equiv 2^{2u+1} - 4 \pmod{24} \equiv 4 \pmod{24},$$

because $a_{j,u}^2$ are odd squares which are not divisible by 3 so they are congruent to $1 \pmod{24}$.

- Then

$$M_u := M - \sum_{j=5}^{k_u} a_{j,u}^2 \equiv 4 \pmod{24},$$

and is a number of size $M - O_T(M^{2/3})$.

- By a result of Ching **2020**, for large M, the number M_u can be written as

$$M_u = a_{1,u}^2 + a_{2,u}^2 + a_{3,u}^2 + a_{4,u}^2,$$

where each prime factor of $a_{i,u}$ exceeds M^ω (where $\omega > 0$ is some fixed small constant).

- Furthermore, the number of such representations is

$$\gg \frac{M}{(\log M)^4}.$$

- Choosing M such that $M^\omega > W$ ensures that $a_{j,u}$ are coprime to P. It remains to ensure that we may assume that

$$a_{1,u} > a_{2,u} > a_{3,u} > a_{4,u} > a_{5,u}.$$

• Well, if the last inequality fails it means that $a_{4,u} \leq M^{1/3}$. Note that $a_{3,u} \leq M^{1/2}$. With $a_{3,u}$ and $a_{4,u}$ fixed, we have that

$$M_u - a_{3,u}^2 - a_{4,u}^2 = a_{1,u}^2 + a_{2,u}^2.$$

• The number of such representation is a divisor function (the number of divisors of the left–hand side in $\mathbb{Z}[i]$) and so it is at most $M^{o(1)}$ as $M \to \infty$. So, the number of such representations $(a_{1,u}, a_{2,u}, a_{3,u}, a_{4,u})$ is

$$\ll M^{1/2+1/3+o(1)} \quad \text{as} \quad M \to \infty,$$

and this is much smaller than $M/(\log M)^4$. Thus, we may ensure that $a_{4,u} > a_{5,u}$.

• To ensure that

$$a_{1,u} > a_{2,u} > a_{3,u} > a_{4,u},$$

we note that the number of representations of M_u as a sum of four non-distinct squares is $\ll M^{1/2+o(1)}$ as $M \to \infty$ and this is much smaller that the number of representations from Ching's theorem.

• Thus, we can indeed find M's which admit representations as in (3) where all $a_{i,u}$ for $u = 1, \ldots, T$ and $i = 1, \ldots, k_u$ are distinct and coprime to P.

• Now it suffices to choose n such that for each $u = 1, \ldots, T$,

$$2^{T-u}n \pm a_{i,u}, \qquad i = 1, \ldots, k_u$$

are consecutive primes.

• To ensure that they are primes, we can use Schinzel's Hypothesis H.

• The only condition we need to check is that for all primes p there is n such that the polynomial

$$f(X) := \prod_{u=1}^{T} \prod_{i=1}^{k_u} (2^{T-u}X - a_{i,u})(2^{T-u}X + a_{i,u})$$

evaluated in n is not a multiple of p.

• This is a condition that needs to be checked only for primes q which are at most the degree of the above polynomial which is W and for such primes we have that $f(n)$ is nonzero modulo q for all $q \le W$ from the way we have chosen the $a_{i,u}$ for $u = 1, \ldots, T$ and $i = 1, \ldots, k_u$.

• The only fact that now needs to be checked is to ensure that for fixed u,

$$2^{T-u}n - a_{1,u}, \ldots, \ 2^{T-u}n - a_{k_u,u}, \ 2^{T-u}n + a_{k_u,u}, \ldots, \ 2^{T-u}n + a_{1,u}$$

are consecutive primes.

• Let $h_u := a_{1,u} - k_u$ be the cardinality of

$$[1, a_{1,u}] \backslash \{a_{k_u,u}, a_{k_u-1,u}, \ldots, a_{1,u}\}, \tag{4}$$

and let $\{b_{1,u}, b_{2,u}, \ldots, b_{h_u,u}\}$ be the elements in the set shown at (4). Now we choose $\mathcal{P}_u$ to be a set or $2h_u$ primes labeled $q_{1,j}, \ q_{-1,j}$ for $j = 1, \ldots, h_u$ which are all larger than W and ask of n to be even and to solve the Chinese Remainder Lemma

$$2^{T-u}n \equiv \varepsilon b_{j,u} \pmod{q_{\varepsilon,j}} \quad \text{for} \quad \varepsilon \in \{-1, 1\}. \tag{5}$$

• We further choose the sets of such primes $\mathcal{P}_u$ to be disjoint as u ranges from 1 to T.

• Then we choose n even in such a way that all Chinese Remainder Lemmas (5) are satisfied for $u = 1, \ldots, T$.

• This puts n into a progression $A \pmod{B}$, where

$$B := 2 \prod_{u=1}^{T} \prod_{q \in \mathcal{P}_u} q.$$

• Finally, we return to our problem and we now only look for n in the arithmetic progression $A \pmod{B}$ such that

$$2^{T-u} n \pm a_{i,u} \quad \text{for} \quad i = 1, \ldots, k_u, \quad \text{and} \quad u = 1, \ldots, T$$

are primes.

• The fact that we can find infinitely many such n is again a consequence of Schinzel's Hypothesis H.

• Clearly, such n's do ensure that

$$2^{T-u}n-a_{1,u},\ldots,\ 2^{T-u}-a_{k_u,u},\ 2^{T-u}n+a_{k_u,u},\ldots,\ 2^{T-u}n+a_{1,u},$$

are consecutive primes just because any intermediary positive integer (positive integer which is between two consecutive members of the above list) is a multiple of q for some $q \in \mathcal{P}_u$.

Comment

• Again by the Bateman-Horn for our construction it predicts that the number of $N \le Y$ such that $f_2(N) \ge T$ is

$$\gg \frac{Y^{1/2}}{(\log Y)^W}.$$

It would be interesting to also make T tend to infinity slowly with N. Perhaps it is true that

$$f_2(N) \gg \log\log N \quad \text{holds for infinitely many} \quad N.$$

This would be consistent with a *Uniform Bateman-Horn conjecture* with an error term of size $O(\sqrt{Y})$. We give no further details.

THANK YOU VERY MUCH!