Sum of consecutive powers as a perfect power

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Historical background

The question of when a sum of consecutive powers is a perfect power has attracted attention over the years. Some milestones:

1875 **Lucas' cannonball problem:** $1^2+2^2+\cdots+N^2 = M^2$ only for $N = 1$ and $N = 24$ ($M = 70$). Proved by Watson (1918); elementary proofs by Ma (1985) and Anglin (1990).

1942 **Ljunggren:** $x^2+1 = 2y^4$ has only $x = 1$ and $x = 239$ (positive solutions), with an intricate proof. Much simpler proof by Steiner–Tzanakis (1991), using linear forms in logarithms.

1956 **Schäffer:** $1^k + 2^k + \cdots + x^k = y^n$ has finitely many solutions for fixed (k, n), apart from four exceptional pairs.

1985 **Cassels:** $(x-1)^3 + x^3 + (x+1)^3$ is a square only for $x = 0, 1, 2, 24$.

2004 **Bennett–Győry–Pintér:** Schäffer's conjecture for $k \leq 11$ and all n.

2010s– **The modular method** enters: Zhang; Bennett–Patel–Siksek; Patel–Siksek. Very recently, Chen–Choi–Houle: Schäffer's conjecture for even $k \leq 42$.

The general problem and known results

General problem

Fix $k \geq 2$ and $d \geq 2$. Determine all integer solutions of

$$x^k + (x+1)^k + \cdots + (x+d-1)^k = y^n, \qquad n \geq 2. \tag{1}$$

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Known cases, as surveyed in the paper:

- $k = 2$, $2 \leq d \leq 10$: Cohn (1996), Zhang (2014), Patel (2018).
- $k = 3$, $2 \leq d \leq 50$: Uchiyama (1979), Cassels (1985), Stroeker (1995), Zhang (2014), Bennett–Patel–Siksek (2017).
- $k = 4$, $d = 2$: Ellenberg (2004), Bennett–Ellenberg–Ng (2010).
- $k = 4, 5, 6$, $d = 3$: Zhang (2014), Bennett–Patel–Siksek (2016).
- **Patel–Siksek (2017):** for even k, (1) has no non-trivial solutions for *almost all* $d \geq 2$.

Two consecutive powers

$$x^k + (x+1)^k = y^n, \qquad n \geq 3. \tag{2}$$

- **Trivial solutions:** $x \in \{-1, 0\}$. For even k the map $x \mapsto -1 - x$ is a symmetry, so we may take $x \geq 1$.

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- $k \equiv 0 \pmod 4$**:** $\left(x^{k/4}\right)^4 + \left((x+1)^{k/2}\right)^2 = y^n$ has signature $(2, 4, n)$, so only trivial solutions (Ellenberg; Bennett–Ellenberg–Ng).

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- k **odd:** open. Already $k = 5$ looks very hard.
- $k \equiv 2 \pmod 4$**:** this is the subject of this talk. Not many results were known before; the trivial solutions obstruct Darmon's program.

Main results

Theorem 1 (K.–Tzanakis)

Let $k \equiv 2 \pmod 4$ with $6 \leq k \leq 100$. Then every solution of (2) has $x \in \{-1, 0\}$.

Theorem 2 (K.–Tzanakis)

Let $k \equiv 2 \pmod 4$, and suppose every odd prime divisor of k is $\equiv 3 \pmod 4$. Then the non-trivial solutions of (2) are exactly

$$(x, y, n, k) \in \{(119, 13, 4, 2),\ (-120, 13, 4, 2)\}.$$

A change of variables

For even k, the symmetry $x \mapsto -1 - x$ lets us assume $x \geq 1$; then $y > 1$ is odd. Write the consecutive integers $x,\ x+1$ as $\frac{x-1}{2},\ \frac{x+1}{2}$, i.e. make the change of variables $x \mapsto \frac{x-1}{2}$ with $x > 1$ odd. Then (2) becomes

$$\frac{(x-1)^k + (x+1)^k}{2} = 2^{k-1} y^n, \qquad x, y \geq 3 \text{ odd}, \quad n \geq 3. \tag{3}$$

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For $k = 2$ the left-hand side is $x^2 + 1$, so (3) becomes

$$x^2 + 1 = 2y^n, \qquad n \geq 3. \tag{4}$$

Theorem 3 (Cohn 1996; Abu Muriefah–Luca–Siksek–Tengely 2009)

The only solutions of (4) are $(|x|, y, n) = (1, 1, n)$ and $(239, 13, 4)$.

In the original variables, $x = 239$ gives $119^2 + 120^2 = 13^4$. Theorem 3 is also a key ingredient for $k \geq 6$.

The key lemmas (I): a factorisation

Let $k \equiv 2 \pmod 4$, and let $g_k(t) := \dfrac{(t-1)^k + (t+1)^k}{2}$ be the left-hand side of (3).

Lemma 1

We have $g_k(t) \in \mathbb{Z}[t]$ and $g_k(t) = (t^2+1)\, f_k(t^2)$, where $f_k(t) \in \mathbb{Z}[t]$ has degree $\frac{k}{2} - 1$ and

$$f_k(t) = \sum_{i=0}^{(k/2-1)/2} 2^{k/2-(2i+1)} \binom{k/2}{2i+1} (t+1)^{2i}\, t^{(k/2-(2i+1))/2}.$$

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Examples. $f_6(t) = t^2 + 14t + 1, \quad f_{10}(t) = t^4 + 44t^3 + 166t^2 + 44t + 1.$

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Hence (3) becomes $(x^2+1)\, f_k(x^2) = 2^{k-1} y^n$. The two factors on the left are almost coprime; this is the starting point of the whole proof.

The key lemmas (II): splitting the equation

Let $k \equiv 2 \pmod 4$, $k \geq 6$, let $\mathcal{P}$ be the set of primes $p \mid \frac{k}{2}$ with $p \equiv 1 \pmod 4$, and let $\mathcal{D}$ be the set of divisors of $\frac{k}{2}$ all of whose prime factors lie in $\mathcal{P}$ (so $1 \in \mathcal{D}$).

Lemma 2

Let (x, y, n) be a solution of (3) with $n > 2v_p(k)$ for every prime $p \equiv 1 \pmod 4$ dividing k. Then there are coprime $d_1, d_2 \in \mathcal{D}$ and coprime odd integers $y_1, y_2 \geq 1$ with $y = y_1 y_2$ such that

$$x^2 + 1 = 2^{\frac{d_2}{d_1}} y_1^n, \tag{5}$$

$$f_k(x^2) = 2^{k-2\frac{d_1}{d_2}} y_2^n. \tag{6}$$

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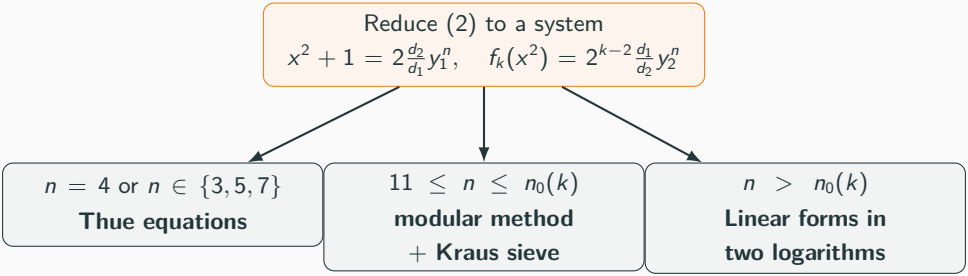
Consequence. If $\frac{k}{2}$ is divisible only by primes $\equiv 3 \pmod 4$, then $\mathcal{P} = \varnothing$, so Lemma 2 gives $d_1 = d_2 = 1$ and (5) becomes $x^2 + 1 = 2y_1^n$. Theorem 3 then gives Theorem 2.

Strategy of the proof

From now on we focus on $6 \leq k \leq 100$ with $\frac{k}{2}$ divisible by a prime $p \equiv 1 \pmod 4$, i.e.

$$k \in \{10, 26, 30, 34, 50, 58, 70, 74, 78, 82, 90\}.$$



Without loss of generality $n = 4$ or n is an odd prime.

Large exponents via linear forms in logarithms

Let $X = x^2 + 1$, so $X = 2\frac{d_2}{d_1} y_1^n$ by (5). Replacing x^2 by $X - 1$ in (6) gives

$$f_k(X-1) = X^{k/2-1} + h_{k/2-2}X^{k/2-2} + \cdots + h_0 = 2^{k-2}\frac{d_1}{d_2}\, y_2^n, \qquad h_{k/2-2} = k\left(\tfrac{k}{2} - 1\right).$$

Dividing by $X^{k/2-1} = \left(2\frac{d_2}{d_1}\right)^{k/2-1} y_1^{n(k/2-1)}$ gives

$$A\left(\frac{y_2}{y_1^{k/2-1}}\right)^n = 1 + \frac{h_{k/2-2}}{X} + \cdots + \frac{h_0}{X^{k/2-1}}, \qquad A = 2^{\frac{k}{2}-1}\left(\frac{d_1}{d_2}\right)^{k/2} \neq 1.$$

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Hence, for $n > n_2$ ($n_2 \leq 554$) and X large, with $\alpha_1 = y_1^{k/2-1}/y_2$, $\alpha_2 = A$:

$$0 < |\Lambda| < \frac{c_1}{y_1^n}, \qquad \Lambda = n\log\alpha_1 - \log\alpha_2, \qquad c_1 \leq 0.50255\, k^2\big(\tfrac{k}{2} - 1\big).$$

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Laurent (2008)

$$\log|\Lambda| \;\ge\; -C_2\left(\max\{\log b' + 0.38,\ m,\ 1\}\right)^2 \log A_1 \log A_2, \qquad b' = \frac{n}{\log A_2} + \frac{1}{\log A_1}.$$

Here $h(\alpha_1) < \log(1.5\, y_1^{k/2-1})$, $h(\alpha_2) \le \frac{k}{2}\log k$, and $(m, C_2) = (10, 25.2)$ or $(12, 23.4)$.

The upper bound $n_0(k)$

Comparing the two bounds for $|\Lambda|$ gives an explicit bound for n.

Proposition 1

If (x, y, n) is a solution of (3) with n prime, then $n \leq n_0(k)$, where:

k	$n_0(k)$	k	$n_0(k)$
10	173 419	70	16 550 269
26	1 438 387	74	18 440 172
30	2 084 701	78	20 987 069
34	2 875 394	82	23 725 616
50	7 627 398	90	29 791 351
58	11 043 693		

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So we only have to resolve the cases $n \leq n_0(k)$: $n \leq 7$ by Thue equations, and $11 \leq n \leq n_0(k)$ by the modular method.

Elliptic curves

An **elliptic curve** $E/\mathbb{Q}$ is given by a long Weierstrass model

$$E\colon\ Y^2 + a_1XY + a_3Y = X^3 + a_2X^2 + a_4X + a_6, \qquad a_i \in \mathbb{Z}, \quad \Delta(E) \neq 0,$$

and has a conductor N_E, divisible exactly by the primes of bad reduction.

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Traces of Frobenius. For a prime ℓ of good reduction,

$$a_\ell(E) = \ell + 1 - \#\tilde{E}(\mathbb{F}_\ell), \qquad |a_\ell(E)| \leq 2\sqrt{\ell}.$$

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Residual representation. For a prime n, the action of $G_\mathbb{Q} = \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ on $E[n] \cong (\mathbb{Z}/n\mathbb{Z})^2$ gives

$$\bar{\rho}_{E,n}\colon G_\mathbb{Q} \longrightarrow \mathrm{GL}_2(\mathbb{F}_n),$$

unramified at every $\ell \nmid nN_E$, with $\operatorname{tr} \bar{\rho}_{E,n}(\mathrm{Frob}_\ell) \equiv a_\ell(E) \pmod{n}$.

Modular forms and modularity

Let $f = \sum_{m \geq 1} a_m(f)\, q^m$ be a newform of weight 2, trivial character and level N, with coefficient field $K_f = \mathbb{Q}\big(a_m(f)\big)$.

Galois representations and modularity

- **Eichler–Shimura:** for each prime $\mathfrak{n} \mid n$ of K_f there is $\bar{\rho}_{f,\mathfrak{n}} \colon G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathbb{F}_{\mathfrak{n}})$, unramified outside nN, with $\operatorname{tr} \bar{\rho}_{f,\mathfrak{n}}(\mathrm{Frob}_\ell) \equiv a_\ell(f) \pmod{\mathfrak{n}}$ for $\ell \nmid nN$.
- **Modularity theorem** (Wiles, Taylor–Wiles, Breuil–Conrad–Diamond–Taylor): every $E/\mathbb{Q}$ is *modular*, which means that there is a newform f of level N_E with $K_f = \mathbb{Q}$ and $a_\ell(f) = a_\ell(E)$ for all $\ell \nmid N_E$.

The modular method: the Frey curve

Let $n \geq 11$ be prime and let (x, y_1) satisfy (5) for a pair (d_1, d_2) with $d_1 d_2 > 1$. Following Bennett–Skinner, we attach to it the **Frey curve**

$$E: \quad Y^2 = X^3 + 2x\,X^2 + 2\tfrac{d_2}{d_1} y_1^n\,X = X^3 + 2x\,X^2 + (x^2+1)\,X.$$

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$$\Delta(E) = -2^8\,\tfrac{d_2^2}{d_1^2}\,y_1^{2n},$$
$$c_4(E) = 2^4(x^2-3),$$
$$N(E) = 2^7 \prod_{p \mid d_1 d_2 y_1} p.$$

- Every odd prime $q \mid y_1$ with $q \nmid d_1 d_2$ has multiplicative reduction, and $n \mid v_q(\Delta)$.
- E is modular.
- $(0,0)$ is a rational 2-torsion point, so $\bar{\rho}_{E,n}$ is irreducible for $n \geq 11$ (Mazur, Kenku).

The modular method: level lowering

Lemma 3 (Ribet's level lowering)

There are a newform $f \in S_2^{\mathrm{new}}(\Gamma_0(N_n))$ of level

$$N_n = 2^7 \prod_{p \mid d_1 d_2} p$$

and a prime $\mathfrak{n} \mid n$ of K_f such that $\bar{\rho}_{E,n} \simeq \bar{\rho}_{f,\mathfrak{n}}$.

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The level N_n does *not* depend on x or y_1. For our values of k, $d_1 d_2$ is a power of a single prime $p \in \{5, 13, 17, 29, 37, 41\}$, so

$$N_n \in \{640,\ 1664,\ 2176,\ 3712,\ 4736,\ 5248\}.$$

Hence there are finitely many candidates f, and they can be listed by computer.

The modular method: eliminating newforms

Proposition 2 (comparing traces)

Suppose $\bar{\rho}_{E,n} \simeq \bar{\rho}_{f,\mathfrak{n}}$ and let $q \nmid 2d_1d_2n$ be prime. Then

$$a_q(f) \equiv a_q(E) \pmod{\mathfrak{n}} \text{ if } q \nmid y_1, \qquad a_q(f) \equiv \pm(q+1) \pmod{\mathfrak{n}} \text{ if } q \mid y_1.$$

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- $a_q(E)$ depends only on x mod q, so it lies in the finite set $S_q = \{a_q(E_{x_0}) : x_0 \in \mathbb{F}_q,\ x_0^2 + 1 \neq 0\}$, where E_{x_0} is E with x replaced by x_0.
- Hence n divides

$$B_q(f) := q \cdot \mathcal{N}_{K_f/\mathbb{Q}}\Big(\big((q+1)^2 - a_q(f)^2\big) \prod_{a \in S_q} \big(a - a_q(f)\big)\Big).$$

- If $B_q(f) \neq 0$ for some q, this bounds n, and f is eliminated for large n. The newforms with $B_q(f) = 0$ for all q survive; they are dealt with later by a Kraus sieve.

The possible pairs (d_1, d_2)

By Lemmas 1 and 2, a solution of (3) gives a solution of the system (5)–(6) for some coprime pair $(d_1, d_2) \in \mathcal{D}^2$. For $6 \leq k \leq 100$, the hypothesis $n > 2v_p(k)$ holds for all $n \geq 5$.

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- $d_1 = d_2 = 1$: then (5) is $x^2 + 1 = 2y_1^n$, which is excluded by Theorem 3.
- If $\mathcal{P} = \varnothing$, this is the only pair; this gives Theorem 2.
- Otherwise $d_1 d_2 > 1$. The table lists the possible pairs for $6 \le k \le 100$.

Pairs (d_1, d_2) with $d_1 d_2 > 1$

k	(d_1, d_2)
10, 30, 70, 90	$(1,5)$, $(5,1)$
26, 78	$(1,13)$, $(13,1)$
34	$(1,17)$, $(17,1)$
50	$(1,5)$, $(1,25)$, $(5,1)$, $(25,1)$
58	$(1,29)$, $(29,1)$
74	$(1,37)$, $(37,1)$
82	$(1,41)$, $(41,1)$

Surviving newforms

We compute all newforms of level N_n and apply Proposition 2 with $q < 50$.

k	N_n	$\dim S_2^{\mathrm{new}}$	surviving f
10, 30, 50, 70, 90	640	16	8 rational
26, 78	1664	48	8 rational
34	2176	64	none
58	3712	112	4 rational; $n = 13$
74	4736	144	none
82	5248	160	4 rational

Proposition 3

For $k = 34, 74$ every newform is eliminated, so $n \leq 7$. For the other k, only rational newforms remain (and $n = 13$ for $k = 58$).

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We compute all newforms of level N_n and apply Proposition 2 with $q < 50$.

k	N_n	$\dim S_2^{\mathrm{new}}$	surviving f
10, 30, 50, 70, 90	640	16	8 rational
26, 78	1664	48	8 rational
34	2176	64	none
58	3712	112	4 rational; $n = 13$
74	4736	144	none
82	5248	160	4 rational

Proposition 3

For $k = 34, 74$ every newform is eliminated, so $n \leq 7$. For the other k, only rational newforms remain (and $n = 13$ for $k = 58$).

Why some newforms cannot be eliminated

(5) alone has solutions with $y_1 = 1$: $3^2 + 1 = 2 \cdot 5$, $7^2 + 1 = 2 \cdot 5^2$, $5^2 + 1 = 2 \cdot 13$, $9^2 + 1 = 2 \cdot 41$. Their Frey curves have conductor $640, 640, 1664, 5248$, and their newforms have $B_q(f) = 0$ for all q. Only (6) rules them out, since it forces $y_1 \geq 3$.

A Kraus sieve using both equations

Fix a prime $n \geq 11$ and $t \geq 2$ with $\ell = tn + 1$ prime and $\ell \nmid k$. Every n-th power mod ℓ lies in $\mu_t(\mathbb{F}_\ell) \cup \{0\}$, so a solution of (5)–(6) gives a point of

$$A(t, \ell) = \left\{ (x_0, \zeta) \in \mathbb{F}_\ell \times \mu_t(\mathbb{F}_\ell) \ : \ \tfrac{d_1}{2d_2}(x_0^2 + 1) = \zeta, \ \ \tfrac{d_2\, f_k(x_0^2)}{2^{k-2} d_1} \in \mu_t(\mathbb{F}_\ell) \cup \{0\} \right\}.$$

Here $E_{x_0,\zeta} \colon Y^2 = X^3 + 2x_0X^2 + 2\frac{d_2}{d_1}\zeta X$ over $\mathbb{F}_\ell$. The second condition encodes (6).

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Proposition 4

Suppose that

$$n \nmid \mathcal{N}_{K_f/\mathbb{Q}}\big(a_\ell(f) - a_\ell(E_{x_0,\zeta})\big) \qquad \text{for all } (x_0,\zeta) \in A(t,\ell).$$

If $\ell \equiv 1 \pmod 4$, suppose also $n \nmid \mathcal{N}_{K_f/\mathbb{Q}}\big(a_\ell(f)^2 - 4\big)$ (the case $\ell \mid y_1$). Then no solution with exponent n arises from f.

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Proposition 5

For $k \neq 34, 74$ and every (d_1, d_2), the system (5)–(6) has no solution with $11 \leq n \leq n_0(k)$.

Computation in Sage with $t \leq 1050$: about 9 hours in total.

Small exponents, $n \in \{3, 4, 5, 7\}$

For $n \in \{3, 4, 5, 7\}$, the problem can be reduced, using elementary number theory, to the resolution of Thue equations, which are solved with PARI/GP.

Proposition 6

For $6 \leq k \leq 100$, equation (3) has no solutions with $n \in \{3, 4, 5, 7\}$.

Open questions and future work

- **Odd k.** The equation $x^k + (x+1)^k = y^n$ is open for odd k; already $k = 5$ seems very challenging.
- **All $k \equiv 2 \pmod 4$.** Theorem 2 covers infinitely many k, and the method should work for each further k separately. But the levels $2^7 \prod_{p \mid d_1 d_2} p$ and the bounds $n_0(k)$ grow with k. Can all $k \equiv 2 \pmod 4$ be treated uniformly?
- **Different exponents.** The equation $x^p + (x+1)^q = y^n$, using Frey hyperelliptic curves and Darmon's program.

Thank you!