The Mordell-Schinzel conjecture for cubic diophantine equations

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April, 2025

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- Jacobsthal (1939) $\deg G = 2$, symmetric in x, y.
- Mordell (1952) Claim: always infinitely many solutions.
  proof details for $xyz = ax^n + by^m + c$, for $|abc| > 1$.

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- Counterexamples in Jacobsthal (1939)
  $xyz = x^2 + y^2 - 1 \pm x \pm y$.
- Schinzel (2015) proof for
  $xyz = ax^n + by^m + c + \text{(other terms)}$
  provided $n, m \geq 3$ and $|abc| > 1$.

Methods

- $\deg G = 2$: Fix $z = d$, then $dxy = G(x, y)$ degree 2,
  Pell-type equations, well known.
- $\deg G = 3$: then $dxy = G(x, y)$ are elliptic curves.
  Lot of data by Szabolcs Tengely.
- Mordell-Schinzel: explicit solutions using recursions.
- (with David Villalobos Paz and Jennifer Li):
  When Mordell-Schinzel fails, the
  automorphism group is infinite.

Why is degree 2 different?

If $\deg G = 2$ then z is invariant under the automorphisms.
The problem breaks up into the independent Pell-type equations $dxy = G(x, y)$.

If $\deg G \geq 3$, then z is not invariant. So a solution of $d_0 xy = G(x, y)$ is transformed into solutions of infinitely many different $d_i xy = G(x, y)$,

Claim: There are no invariant polynomials, so the solutions are Zariski dense.

János Kollár, *Log K3 surfaces with irreducible boundary*, https://arxiv.org/abs/2407.08051.

János Kollár and David Villalobos-Paz, *Cubic surfaces with infinite, discrete automorphism group*, https://arxiv.org/abs/2410.03934.

János Kollár and Jennifer Li, *The Mordell-Schinzel conjecture for cubic surfaces*, https://arxiv.org/abs/2412.12080.

Standard form

- the terms in G that are divisible by xy can be absorbed into xyz.
- For $abc \neq 0$, write equation as

$$xyz = ax^n + by^m + c + \sum_{i=1}^{n-1} a_i x^i + \sum_{j=1}^{m-1} b_j y^j$$

$$A(x) := ax^n + c + \sum_{i=1}^{n-1} a_i x^i, B(y) := by^m + c + \sum_{j=1}^{m-1} b_j y^j.$$

$$xyz = A(x) + B(y) - c,$$

Warning: c appears 3 times.

$$S_{A,B} := \big(xyz = A(x) + B(y) - c\big) \subset \mathbb{A}^3_{\mathbb{Z}}.$$

Assume from now on that $abc \neq 0$.

Mordell-Schinzel conjecture

Conjecture

If $m, n \geq 3$ *(and* $abc \neq 0$*) then*

$$xyz = ax^n + by^m + c + \sum_{i=1}^{n-1} a_i x^i + \sum_{j=1}^{m-1} b_j y^j \qquad (MS1)$$

has infinitely many integer solutions.

For historical accuracy note that:
— Neither author stated this as a conjecture.
— Mordell claimed it as true.
— Schinzel proved it when $|abc| > 1$.

Finding automorphisms I

Theorem (with David Villalobos Paz)

If $n, m \geq 3$ and $|abc| = 1$ then the automorphism group of $S_{A,B}$ is infinite.

Corollary of proof:

There is a polynomial $P(A, B)$ in the coeffs of A, B such that

— either all solutions satisfy $|xyz| < P(A, B)$,

— or there are infinitely many solutions.

Problem: The automorphisms are very complicated.

Finding automorphisms II

Example

Set $T_{n,m} := (xyz = x^3 + y^3 + dx + ey + 1)$ for $d, e \in \mathbb{Z} \setminus \{0\}$. Then

$$\mathrm{Aut}_{\mathbb{Z}}(T_{n,m}) = \mathrm{Aut}_{\mathbb{C}}(T_{n,m}) \cong \begin{cases} \mathbb{Z} \text{ if } d \neq e, & \text{and} \\ \mathbb{Z} \rtimes \mathbb{Z}/2 \text{ if } d = e. \end{cases}$$

In their simplest form, the coordinate functions of the generator of $\mathbb{Z}$ are given by polynomials of

- degrees 13, 34, and 55, containing
- 110, 998, and 2881 monomials.

Cubic case of the Mordell-Schinzel conjecture I

Theorem (with Jennifer Li)

For every $a, b \in \mathbb{Z} \setminus \{0\}$ and $a_1, a_2, b_1, b_2, c \in \mathbb{Z}$, the cubic equation

$$xyz = ax^3 + by^3 + c + a_2x^2 + a_1x + b_2y^2 + b_1y$$

has infinitely many integral solutions.

Plan of the proof:

- If $c = 0$ then trivial solutions $(0, 0, z)$.
- If $|abc| > 1$: done by Schinzel.
- In remaining cases, normal form:

$$S_{a_1,a_2,b_1,b_2} := \left(xyz = x^3 + y^3 + 1 + a_2x^2 + a_1x + b_2y^2 + b_1y\right)$$

Cubic case of the Mordell-Schinzel conjecture II

$$S_{a_1,a_2,b_1,b_2} := \left(xyz = x^3 + y^3 + 1 + a_2x^2 + a_1x + b_2y^2 + b_1y\right)$$

- Trivial solutions: (x_0, y_0, z_0), where $x_0 = \pm 1, y_0 = \pm 1$.
- Step 1: If the orbits of the trivial solutions is finite, then $|a_i|, |b_j|$ are bounded.
- Step 2: For small $|a_i|, |b_j|$, computer search for solution with $|x_0y_0|$ large enough.

Automorphisms of cubic surfaces I

$S = \big(f(x,y,z) = 0\big) \subset \mathbb{A}^3$ cubic surface.

Problem: Determine $\text{Aut}(S)$.

Note: over $\mathbb{Z}$, or $\mathbb{R}$ or $\mathbb{C}$ .....

Well understood: Linear automorphisms:

Subgroups of $\text{Aut}^{\text{lin}}(\mathbb{A}^3) \cong \mathbb{A}^3 \rtimes \mathrm{G}L_3$.

Example. $\text{Aut}_{\mathbb{C}}(xyz = 1) \cong (\mathbb{C}^\times)^2 \rtimes S_3$.

Automorphisms of cubic surfaces I

Markov/Markoff (1879)

$3xyz = x^2 + y^2 + z^2$.

Interchange the roots of $x^2 - (3yz)x + (y^2 + z^2) = 0$:

$\tau_x : (x, y, z) \mapsto (-x + 3yz, y, z)$

Similarly: τ_y, τ_z.

Theorem. $\langle \tau_x, \tau_y, \tau_z \rangle$ is the free product $\mathbb{Z}/2 * \mathbb{Z}/2 * \mathbb{Z}/2$, so infinite.

No other cubic surface with **infinite discrete** automorphism group was known.

But: it was **rediscovered** many times.

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Ruzsa lecture on July 2024: $(x^2 + 1)(y^2 + 1) = z^2 + 1$.

with $t = z - xy$ becomes a Markov equation:

$2xyt = x^2 + y^2 - t^2$.

Automorphisms of cubic surfaces II

Write $A(x) = xA^*(x) + c$. Then the equation becomes

$$xyz = xA^*(x) + B(y), \quad \text{equivalently}$$
$$x\big(yz - A^*(x)\big) = B(y).$$

Plan:

$$\sigma_y : (x, y) \mapsto \big(yz - A^*(x), y\big) = \left(\frac{B(y)}{x}, y\right).$$

Problem: What happens to z?

Automorphisms of cubic surfaces III

Proposition. Set

$$\bar{A}(t) := \frac{1}{a_0^{n-1}a_n} t^n A\left(\frac{a_0}{t}\right), \quad \text{then } \sigma_y \text{ maps}$$

the surface $S_{A,B} := \left(xyz = A(x) + B(y) - c\right)$

to the surface $S_{\bar{A},B} := \left(xyz = \bar{A}(x) + B(y) - c\right)$.

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Corollary

We get $\mathbb{Z}$-isomorphisms only when $|abc| = 1$.

Automorphisms of cubic surfaces IV

$$A(t) := t^3 + a_2 t^2 + a_1 t + 1, \quad B(t) := t^3 + b_2 t^2 + b_1 t + 1,$$
$$\bar{A}(t) := t^3 + a_1 t^2 + a_2 t + 1, \quad \bar{B}(t) := t^3 + b_1 t^2 + b_2 t + 1.$$

For a given A, B, there are 8 **companion** surfaces in play:

$$S_{A,B}, S_{\bar{A},B}, S_{A,\bar{B}}, S_{\bar{A},\bar{B}}, S_{B,A}, S_{\bar{B},A}, S_{B,\bar{A}}, S_{\bar{B},\bar{A}}.$$

Theorem (with Villalobos Paz)

The groupoid of isomorphisms between all the surfaces $S_{A,B}$ is generated by

1. *linear isomorphisms,*
2. $\sigma_y : S_{A,B} \cong S_{\bar{A},B}$
3. $\sigma_x : S_{A,B} \cong S_{A,\bar{B}}$.

Automorphisms of cubic surfaces V

$$\begin{array}{ccc} S_{A,B} & \overset{\sigma_x}{\longleftrightarrow} & S_{A,\bar{B}} \\ \sigma_y \updownarrow & & \updownarrow \sigma_y \\ S_{\bar{A},B} & \overset{\sigma_x}{\longleftrightarrow} & S_{\bar{A},\bar{B}} \end{array}$$

Theorem (with Villalobos Paz)

The composite

$$\sigma_{A,B} : S_{A,B} \xrightarrow{\sigma_x} S_{\bar{A},B} \xrightarrow{\sigma_y} S_{\bar{A},\bar{B}} \xrightarrow{\sigma_x} S_{A,\bar{B}} \xrightarrow{\sigma_y} S_{A,B}.$$

generates an infinite, cyclic subgroup of finite index in $\mathrm{Aut}_{\mathbb{Z}}(S_{A,B})$.

Finding solutions I

$$S := \left(xyz = \pm x^n \pm y^m \pm 1 + \sum_{i=1}^{n-1} a_i x^i + \sum_{j=1}^{m-1} b_j y^j\right).$$

$$\sigma_y : (x, y) \mapsto \big(B(y)/x, y\big) \text{ and } \sigma_x : (x, y) \mapsto \big(x, A(x)/y\big);$$

Elementary estimates:

Lemma

Let $p_0 = (x_0, y_0, z_0) \in S$ be a complex point. If

$$\max\{|x_0|, |y_0|\} > 1 + \max\left\{1, \sum_{i=0}^{n-1} |a_i|, \sum_{j=0}^{m-1} |b_j|\right\},$$

then the $\sigma_{A,B}$-orbit of p_0 is infinite.

Finding solutions II

Now assume $n = m = 3$, so S is a cubic surface.

Proposition

The $\mathrm{Aut}_{\mathbb{Z}}(S)$-orbit of every trivial solution is finite iff S is a companion surface of:

$$\begin{aligned} xyz - x^3 - y^3 - 1 &= -x^2 - y^2, \\ xyz - x^3 - y^3 - 1 &= -2x^2 - x - 2y^2 - y, \\ xyz - x^3 - y^3 - 1 &= -2x^2 - x - y^2. \end{aligned}$$

Finding solutions III

$$\begin{array}{lcl} (-7,-17,-47) & : & -x^2 - y^2, \\ (293,-601,1095) & : & -2x^2 - x - 2y^2 - y, \\ (11,-13,9) & : & -2x^2 - x - y^2. \end{array}$$

Finding solutions: higher degrees

Why cubics?

$$(x, y) \mapsto \Big((y^3 + b_2 y^2 + b_1 y + 1)/x, y\Big).$$

Can chose $\pm y$ such that $b_2 y^2$ and $b_1 y$ have same sign.
In degrees ≥ 4, the b_i may cancel each other.

Example. If $A(x), B(y)$ are of the form

$$t^4 - t^2 + 1 + r(t^3 - t)$$

the the orbit of all trivial solutions is finite.

The Mordell-Schinzel method

Preliminary: If $|abc| > 1$, then there are infinitely many $\mathbb{Z}[(abc)^{-1}]$-integral solutions:
x, y monomials in a, b, c, and $z = G(x, y)/(xy)$.

Constructing solutions. Follow the denominator changes:
For every $r > 0$ there is a monomial point p_r such that $\sigma^r_{A,B}(p_r)$ is a $\mathbb{Z}$-integral point.
The x_r, y_r are given as $a^{\lambda_r} b^{\mu_r} c^{\nu_r}$, where λ_r, μ_r, ν_r satisfy Fibonacci-type recursions. (Formula 30 in Schinzel).

Different r give different solutions if $|abc| > 1$.
This is the harder part of Schinzel's papers, very careful estimates are needed.