Number of solutions to a special type of unit equations in two unknowns

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Plan of the talk

Joint work with T. Miyazaki

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- Introduction

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- Brief summary of earlier results

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- Main result
- Sketch of the proof of the main result

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- Gelfond (1940): the first effective finiteness result for the solutions (x, y, z) of (1).

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- Method: Baker's theory of linear forms in complex and p-adic logarithms of algebraic numbers.

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• Let $\mathbb{K}$ be a number field and let $\alpha_1, \alpha_2 \in \mathbb{K} \setminus \{0\}$ be fixed. Consider the S-unit equation

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Theorem (Evertse, Győry, Stewart, Tijdeman, 1988)

Apart from finitely many S-equivalence classes equation (s2) *has at most two solutions.*

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Theorem (Hirata-Kohno, 2006)

$N(a, b, c) \leq 2^{36}$.

- This is the first absolute upper bound for $N(a, b, c)$.

Some solutions of (1)

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$$5+2^2=3^2,\ 5^2+2=3^3; 7+2=3^2,\ 7^2+2^5=3^4;$$
$$3^2+2=11,\ 3+2^3=11; 3^3+2^3=35,\ 3+2^5=35;$$
$$3^5+2^4=259,\ 3+2^8=259; 5^3+2^3=133,\ 5+2^7=133;$$
$$3+10=13,\ 3^7+10=13^3; 89+2=91,\ 89+2^{13}=91^2;$$
$$2+3=5,\ 2^4+3^2=5^2; 91^2+2=8283,\ 91+2^{13}=8283;$$
$$3+13=2^4,\ 3^5+13=2^8; 3+13^3=2200,\ 3^7+13=2200;$$

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$2^1+(2^k-1)^1=(2^k+1)^1,\ 2^{k+2}+(2^k-1)^2=(2^k+1)^2$, where k is any integer with $k \ge 2$.

Some special cases of (1)

- If in (1) we have $x = 1$ then (1) becomes

$$c^z - b^y = a. \tag{2}$$

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- It is well known by a result of Mihăilescu (2004) that if $a = 1$ then the only solution of (2) is $(c, b, z, y) = (3, 2, 2, 3)$. For $a \geq 2$ the conjecture is still open.

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Theorem (Bennett, 2001)

If a, b, c are positive integers with $\min\{c, b\} > 1$ then equation (2) has at most two solutions in positive integers (z, y).

Some special cases of (1)

• The proof of Bennett uses lower bounds for linear forms in logarithms of two algebraic numbers together with a ‘gap principle’ based upon some nice congruence arguments which gives rise to a large gap among three hypothetical solutions.

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• This result is essentially sharp in the sense that there are a number of examples where there are two solutions to equation (2).

• It should be also remarked that the non-coprimality case, i.e. $\gcd(c, b) > 1$ is handled just by a short elementary observation.

Some results of Scott and Styer on equation (1)

• For a, b, c fixed co-prime positive integers with $\min\{a, b, c\} > 1$ consider equation

$$a^x + b^y = c^z \qquad (1)$$

in positive integer unknowns (x, y, z).

Theorem (Scott, 1993)

If $c = 2$ then equation (1) has at most one solution except for (taking $a < b$): $(a, b, c) = (3, 5, 2)$ which has exactly three solutions and $(a, b, c) = (3, 13, 2)$ which has exactly two solutions.

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- The method of the proof is elementary and uses the theory of quadratic fields.
- Note that this method works only for c odd.
- Therefore for c even other method is needed.

Some results of Hu and Le on equation (1)

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• Elementary methods including congruences modulo powers of a single base number together with the theory of continued fractions show that there is a large gap among three hypothetical solutions.

• Then the combination of this gap principle with Baker's method implies that (1) has at most two solutions whenever $\max\{a, b, c\}$ is sufficiently large.

Main result

- For a, b, c fixed co-prime positive integers with $\min\{a, b, c\} > 1$ consider equation

$$a^x + b^y = c^z \tag{1}$$

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Equation (1) has at most two solutions except when $(a, b, c) = (3, 5, 2)$ or $(5, 3, 2)$ which exactly gives three solutions.

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Theorem (Miyazaki and Pink, 202?)

Equation (1) has at most two solutions except when $(a, b, c) = (3, 5, 2)$ or $(5, 3, 2)$ which exactly gives three solutions.

- The exceptional case comes from the identities $3 + 5 = 2^3; 3^3 + 5 = 2^5$ and $3 + 5^3 = 2^7$

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• Our result is essentially sharp and definitive in the sense that there are infinitely many examples allowing equation (1) to have two solutions.

• $2^1 + (2^k - 1)^1 = (2^k + 1)^1,\ 2^{k+2} + (2^k - 1)^2 = (2^k + 1)^2$, where k is any integer with $k \geq 2$.

Main steps of the proof

- The proof of our theorem proceeds under the assumption that $\max\{a, b, c\} < 10^{62}$ and c is even, and there are four main steps.

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- Assume that equation (1) has three solutions (x_t, y_t, z_t), $t = 1, 2, 3$ with $z_1 \le z_2 \le z_3$, that is we have the system

$$a^{x_1} + b^{y_1} = c^{z_1}, \quad a^{x_2} + b^{y_2} = c^{z_2}, \quad a^{x_3} + b^{y_3} = c^{z_3}. \qquad (3)$$

Main steps of the proof

- **Step 2** Under assumption (3) the second step is to find sharp upper bounds for all the exponential unknowns of the first two equations in (3). Namely, we obtain sharp upper bounds for $\max\{z_1, z_2\}$ and $\max\{x_1, y_1, x_2, y_2\}$.

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- This is done elementarily by basically comparing the 2-adic valuations of both sides of each of the three equations occurring in (3).

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- **Step 3** Improving the gap principle of Hu and Le.

- Under assumption (3) the main idea to improve the gap principle of Hu and Le is to consider two congruences 'simultaneously' by using modulus of each powers of the base numbers.

- Under assumption (3) we worked out the above three steps and combined them with several other number theoretical methods (e.g. ternary equations of various signatures) to obtain sharp upper bounds for all letters $a, b, c, x_1, y_1, z_1, x_2, y_2, z_2$ occurring in the first two equations of (3).

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- At this point it is worth noting that although the derived general bounds for all letters in the first two equations of (3) are relatively sharp, a direct enumeration of the solutions impossible.

- Therefore, we worked very carefully and found efficient methods for solving the system

$$a^{x_1} + b^{y_1} = c^{z_1}, \quad a^{x_2} + b^{y_2} = c^{z_2}. \tag{4}$$

in a reasonable computational time.

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$$\begin{cases} \text{none of } a, b, c \text{ is a power}, \quad a, b, c \text{ are pairwise coprime;} \\ a \equiv -1 \pmod 4 \quad \text{or} \quad b \equiv -1 \pmod 4; \\ \max\{a, b\} \geq 11, \quad 18 \leq \max\{a, b, c\} \leq 10^{62}; \\ 2 \mid c, \ \ c > 2. \end{cases} \qquad (*)$$

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- Put $\alpha = \min\{\nu_2(a^2-1)-1, \nu_2(b^2-1)-1\}, \quad \beta = \nu_2(c)$.

Improved bound for z in (1)

Since c is even and $z > 1$, it follows from equation (1) that $a^x + b^y \equiv 0 \pmod 4$. Therefore, one of the following cases holds.

$$\begin{cases} a \equiv 1,\ b \equiv -1 \pmod 4, & 2 \nmid y; \\ a \equiv -1,\ b \equiv 1 \pmod 4, & 2 \nmid x; \\ a \equiv b \equiv -1 \pmod 4, & x \not\equiv y \pmod 2. \end{cases} \tag{5}$$

Put $\Lambda = a^x + b^y$. Since $\Lambda = c^z$, we have

$$z = \frac{1}{\beta} \cdot \nu_2(\Lambda). \tag{6}$$

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Put $\Lambda = a^x + b^y$. Since $\Lambda = c^z$, we have

$$z = \frac{1}{\beta} \cdot \nu_2(\Lambda). \tag{6}$$

• A careful application of a result of Bugeaud on Λ yields a non-trivial upper bound for $\nu_2(\Lambda)$.

Improved bound for z in (1)

Lemma 1 Assume that $\max\{a, b\} \geq 9$. Put $\alpha = \min\{\nu_2(a^2-1)-1,\, \nu_2(b^2-1)-1\}, \quad \beta = \nu_2(c)$. Let (x, y, z) be a solution of equation (1) with $z > 1$. Then

$$z < \mathcal{H}_{\alpha,\beta,m_2}(c; a, b) := \max\{c_1,\, c_2 \log_*^2(c_3 \log c)\}\, (\log a) \log b,$$

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$$z < \mathcal{H}_{\alpha,\beta,m_2}(c; a, b) := \max\{c_1, c_2 \log_*^2(c_3 \log c)\} (\log a) \log b,$$

where

$$(c_1, c_2, c_3) = \begin{cases} \left(\frac{1803.3m_2}{\beta}, \frac{23.865m_2}{\beta}, \frac{143.75(m_2+1)}{\beta}\right), & \text{if } \alpha = 2, \\ \left(\frac{2705m_3}{\alpha\beta}, \frac{156.39m_3\left(1+\frac{\log v_\alpha}{v_\alpha-1}\right)^2}{\alpha^3\beta}, \frac{646.9(m_3+1)}{\alpha^2\beta}\right), & \text{if } \alpha \geq 3 \end{cases}$$

with $v_\alpha = 3\alpha \log 2 - \log(3\alpha \log 2)$, and

$$m_2 = \begin{cases} \frac{\log 8}{\log \min\{a,b\}}, & \text{if } \min\{a, b\} \leq 7, \\ 1, & \text{if } \min\{a, b\} > 7, \end{cases} \qquad m_3 = \frac{\log 2^\alpha}{\log(2^\alpha - 1)}.$$

Improved bound for z in (1)

Corollary

Assume that $\max\{a, b\} \geq 9$ *and put* $M = \max\{a, b, c\}$. *Let* (x, y, z) *be a solution of equation* (1) *with* $z > 1$. *Then*

$$\max\{x, y, z\} < \max\left\{1804 \log^2 M, 46 \log^2 M \log_*^2(416 \log M)\right\}$$

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- The bound in the paper of Hu and Le (2019) paper was

$$\max\{x, y, z\} < 6500 \log^3 M.$$

Sharp bounds for the solutions of the first two equations of (3)

- The key ingredient is the following elementary lemma.

Lemma

Let $(x, y, z) = (X, Y, Z), (X', Y', Z')$ *be two solutions of equation* (1). *Then* $XY' \neq X'Y$, *and*

$$\beta \cdot \min\{Z, Z'\} \leq \alpha + \nu_2(XY' - X'Y).$$

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- As a consequence we get

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Assume that equation (1) *has three solutions* (x_t, y_t, z_t), $t = 1, 2, 3$ *with* $z_1 \le z_2 \le z_3$. *Then*

$$\beta z_t \le \alpha + \nu_2(x_t y_{t+1} - x_{t+1} y_t) \quad (t = 1, 2); \tag{7}$$

$$z_3 < \mathcal{H}_{\alpha,\beta,m_2}(c; a, b), \tag{8}$$

respectively, where

$$\mathcal{H}_{\alpha,\beta,m_2}(u; v, w) := \max\{c_1,\ c_2 \log_*^2(c_3 \log u)\} \cdot \log v \cdot \log w.$$

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Lemma

We have $z_1 \leq z_2 \leq 230$ *and* $\max\{x_1, y_1, x_2, y_2\} < 4300$.

- A direct application of the general Baker type bound gives only

$$\max_{i=1,2}\{x_i, y_i, z_i\} < 1.14 \cdot 10^8.$$

Some notations

• For fixed a, b, c co-prime positive integers with $\min\{a, b, c\} > 1$ equation

$$a^x + b^y = c^z \tag{1}$$

in positive integer unknowns (x, y, z) can be rewritten as

$$A^X + \lambda B^Y = C^Z, \tag{7}$$

where $\lambda \in \{1, -1\}$,
$(A, B, C, \lambda) \in \{(a, b, c, 1), (c, b, a, -1), (c, a, b, -1)\}$ and
$(X, Y, Z) \in \{(x, y, z), (z, y, x), (z, x, y)\}$ is the corresponding permutation of (x, y, z).

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- Thus for every $(A, B, C, \lambda) \in \{(a, b, c, 1), (c, b, a, -1), (c, a, b, -1)\}$ the number of solutions of (1) and (7) are equal.
- We present our improved gap principle obtained for (7) and then we apply it many times in our proof concerning (1).
- For our purpose, it suffices to consider equation (7) under the following conditions (corresponding to $(*)$):

$$\begin{cases} \text{none of } A, B, C \text{ is a power;} \\ 2 \mid C,\ C > 2,\ \max\{A, B\} \geq 11, \quad \text{if } \lambda = 1; \\ 2 \mid A,\ A > 2,\ \max\{B, C\} \geq 11, \quad \text{if } \lambda = -1. \end{cases} \qquad (**)$$

Improved gap principle 1

Proposition (Gap principle 1)

Suppose that equation (7) has three solutions $(X, Y, Z) = (X_r, Y_r, Z_r)$ *with* $r \in \{1, 2, 3\}$ *such that* $Z_1 < Z_2 \leq Z_3$. *and put* $G_2 = \gcd(X_2, Y_2)$. *If* $C^{Z_1} > 2$ *then*

$$C^{Z_2 - Z_1} \mid G_2 \cdot (X_2 Y_3 - X_3 Y_2); \tag{9}$$

Moreover, if either $\lambda = 1$, *or* $\lambda = -1$ *with* $G_2 > 1$, *then*

$$C^{Z_2 - Z_1} < \mathcal{K} \cdot t_{A,B} \cdot \frac{Z_2}{Z_1} \cdot |X_2 Y_3 - X_3 Y_2|, \tag{10}$$

where $t_{A,B} := \frac{\log \min\{A,B\}}{\log \max\{A,B\}}$ *and* $\mathcal{K} \leq \frac{G_2}{G_2 - 1}$.

Remarks

- In the corresponding result of Hu and Le (2019) one has $C^{Z_2-Z_1} \mid Y_2 \cdot (X_2Y_3 - X_3Y_2)$ leading to

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- If $C = \max\{A, B, C\}$ then our improved Baker type bound provides a bound for $|X_2Y_3 - X_3Y_2|$ is logarithmic in C => inequality (10) leads to an improved bound for $C^{Z_2-Z_1}$ and hence for C and $Z_2 - Z_1$, as well.

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- In order to obtain bounds for C and $Z_2 - Z_1$ as sharp as possible, we can iterate the use of inequality (10).

Improved gap principle 2

Proposition (Gap principle 2)

Suppose that equation (7) for $\lambda = 1$ has three solutions $(X, Y, Z) = (X_r, Y_r, Z_r)$ with $r \in \{1, 2, 3\}$ such that $Z_1 = Z_2 < Z_3$. Then one of the following inequalities holds.

$$C^{Z_2/2} < \frac{2}{\log \min\{A, B\}} Z_3,$$

$$C^{Z_2/2}/Z_2 < \max_{t \in \{1,2\}} \left\{ |X_3 Z_2 - X_t Z_3|, \ |Y_3 Z_2 - Y_t Z_3| \right\}.$$

Method of the proof

- Assume that equation (1) has three solutions (x_t, y_t, z_t), $t = 1, 2, 3$ with $z_1 \leq z_2 \leq z_3$, that is we have the system

$$a^{x_1} + b^{y_1} = c^{z_1}, \quad a^{x_2} + b^{y_2} = c^{z_2}, \quad a^{x_3} + b^{y_3} = c^{z_3}. \qquad (3)$$

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- Based upon $z_1 \le z_2 \le z_3$, let (i, j, k) and (l, m, n) be permutations of $\{1, 2, 3\}$ such that

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- Also, define non-negative integers d_z, d_x, d_y and positive integers g_2, g_x, g_y as follows:

$$\begin{aligned} d_z &:= z_2 - z_1, & d_x &:= x_j - x_i, & d_y &:= y_m - y_l, \\ g_2 &:= \gcd(x_2, y_2), & g_x &:= \gcd(y_j, z_j), & g_y &:= \gcd(x_m, z_m). \end{aligned}$$

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- W.l.o.g we may assume that $c > a > b$.
- We have the following uniform lower bounds for a, b, c:

$$a_0 = \max\{11, 2^{\alpha} + 1\}, \; b_0 = 2^{\alpha} - 1, \; c_0 = \max\{18, 3 \cdot 2^{\beta}, 2^{\alpha} + 2\}.$$

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- By applying our improved "Gap principle 1" to (3) with $c = \max\{a, b, c\}$ and $z_1 < z_2$, we get

$$c^{d_z} < \min\left\{2^{\alpha-\beta z_1}\frac{({g_2}')^2}{g_2}, \frac{\log c}{\log(c-1)}\frac{z_2}{z_1}\right\} \cdot z_2\mathcal{H}(c; c, c), \qquad (11)$$

where ${g_2}' = \gcd(c^{d_z}, g_2)$ and $d_z = z_2 - z_1$.

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- If $g_2 \geq 11$ we use the following general inequality

$$c^{d_z} < \min\left\{ \frac{\log c}{\log a_0} 2^{\alpha - \beta z_1}, \frac{\log c}{\log(c-1)} \frac{1}{z_1} \right\} \cdot z_2^2 \mathcal{H}(c; c, c). \quad (12)$$

valid for each g_2 which is a consequence of (11).

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- Since $z_2 = z_1 + d_z$ and d_z is bounded we use the elementary result presented in the third subsection to derive for each given β, α, M_c and g_2 a sharp upper bound for $z_1 \leq U_1$ and also for $z_2 \leq U_1 + d_z$.

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- For each given $(g_2, d_z, \beta, \alpha, z_1, z_2)$ we use inequalities (11) or (12) with $c \leq M_c$ to obtain a new improved bound for c.

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- We iterate the above process.

- As a result we find a list of finitely many possible tuples $(d_z, \beta, \alpha, z_2)$ with the corresponding upper bound for c.

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Proposition (Sharp bounds)

Assume that $d_z > 0, \quad c^{z_1} \equiv 0 \mod 4, \quad c > \max\{a, b\}$.

(i) *Suppose that* $g_2 = 1$. *Then*

$$[\beta, \alpha, z_2, d_z] \leq [10, 18, 19, 4], \quad c < 1.5 \cdot 10^6.$$

(ii) *Suppose that* $g_2 > 1$. *Then*

$$[\beta, \alpha, z_2, d_z] \leq [10, 19, 23, 4], \quad c < 3.4 \cdot 10^6.$$

(iii) *If* $d_z > 1$ *then* $c < 1000$.

Efficient sieve

- Consider the system formed by the first two equations of (3), that is

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• Although the above Proposition provides sharp upper bounds for z_1, z_2 and middle-sized bounds for $c = \max\{a, b, c\}$, a direct enumeration of the solutions of (13) (a kind of brute force search) is impossible.

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• Under the hypothesis of the above Proposition if $c < 1000$ or $(z_1, z_2) \in \{(1, 2), (2, 3)\} => $ system (13) has no solutions.

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- We may suppose that $c \geq c_1 := \max\{1000, c_0\},\ d_z = 1,\ z_2 \geq 4$.
- Elementary arguments together with the bounds presented in the third subsection imply restrictions on $x_1, y_1, z_1, x_2, y_2, z_2$ (divisibility conditions, some inequalities).
- In particular we may suppose that $x_1 < x_2$ or $y_1 < y_2$.

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Lemma

Under the hypothesis of Proposition (Sharp bounds), if system (13) *has a solution* $(x_1, y_1, z_1, x_2, y_2, z_2)$, *then*

$$\min\{a^{x_1}, b^{y_1}\} \geq c, \quad \min\{a^{x_2}, b^{y_2}\} \geq c^2.$$

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- First, we illustrate the method to show that $a^{x_1} \geq c$.
- Suppose on the contrary that $a^{x_1} < c$.

If $y_1 \leq z_1$, then $c > a^{x_1} = c^{z_1} - b^{y_1} \geq c^{z_1} - b^{z_1} > c^{z_1 - 1}$, so $z_1 < 2$, which is absurd as $z_1 \geq 3$.

Efficient sieve

Thus $y_1 > z_1$. On the other hand, from 1st equation, observe that

$$0 < c^{z_1/2} - b^{y_1/2} = \frac{a^{x_1}}{c^{z_1/2} + b^{y_1/2}} < \frac{c}{c^{z_1/2}} < 1.$$

Thus

$$\lceil b^{y_1/z_1} \rceil =: c_2 \le c \le c_3 := \left\lfloor (1 + b^{y_1/2})^{2/z_1} \right\rfloor.$$

Since $y_1 > z_1$, it happens very often that $c_2 > c_3$ for given b, y_1 and z_1.

Efficient sieve

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Efficient sieve

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- For each such tuple and for each possible tuple $(b, c, x_1, y_1, x_2, y_2)$ satisfying

$$z_1 < y_1 \leq \left\lfloor \frac{\log c_u}{\log b_0} z_1 \right\rfloor, \; b_0 \leq b \leq \left\lfloor {c_u}^{z_1/y_1} \right\rfloor,$$

$$\max\{c_1, c_2\} \leq c \leq \min\{c_3, c_u\}, y_2 \leq \left\lfloor \frac{\log c}{\log b} z_2 \right\rfloor,$$

$$x_1 \leq \left\lfloor \frac{\log c}{\log a_0} z_1 \right\rfloor, \quad x_2 \leq \left\lfloor \frac{\log c}{\log a_0} z_2 \right\rfloor$$

Efficient sieve

with $z_2 = z_1 + 1$, we check that equation

$$(c^{z_1} - b^{y_1})^{x_2} = (c^{z_2} - b^{y_2})^{x_1}$$

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- Thus the inequality $a^{x_1} \geq c$ holds.

- The remaining inequalities can be shown exactly in the same way by changing the roles of a, b and z_1, z_2, respectively.

Efficient sieve

Lemma

Under the hypothesis of Proposition (Sharp bounds), if system (13) has a solution $(x_1, y_1, z_1, x_2, y_2, z_2)$, *then* $x_1 \geq x_2$.

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Lemma

Under the hypothesis of Proposition (Sharp bounds), if system (13) has a solution $(x_1, y_1, z_1, x_2, y_2, z_2)$, *then* $x_1 \geq x_2$.

- Suppose that $x_1 < x_2$.

Efficient sieve

Lemma

Under the hypothesis of Proposition (Sharp bounds), if system (13) has a solution $(x_1, y_1, z_1, x_2, y_2, z_2)$, *then* $x_1 \geq x_2$.

- Suppose that $x_1 < x_2$.
- The case where $y_1 < y_2$ can be ruled out by (13) and the previous lemma.

Efficient sieve

- Second, consider the case where $y_1 \geq y_2$. System (13) with $z_2 = z_1 + 1$ implies

$$\frac{1}{\left(1 + 1/a^{x_1}\right)^{1/x_2}} \cdot b^{\frac{y_1 z_2}{x_2 z_1}} < a < \left(1 + c/b^{y_2}\right)^{z_2/(x_2 z_1)} \cdot b^{\frac{y_1 z_2}{x_2 z_1}}.$$

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- Since $b < a < c$, and $c^2 < b^{y_2}$ by the previous Lemma, it follows that

$$\frac{1}{\left(1 + 1/a_1^{x_1}\right)^{1/x_2}} \cdot b^{\frac{y_1 z_2}{x_2 z_1}} < a < \left(1 + 1/c_2\right)^{z_2/(x_2 z_1)} \cdot b^{\frac{y_1 z_2}{x_2 z_1}}, \qquad (14)$$

where $a_1 = \max\{a_0, b+2\}$ and $c_2 = \max\{c_1, b+2\}$.

Efficient sieve

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- For each element in *clist* we use the previous Lemmas to sieve considerably the possible solutions $[x_1, y_1, z_1, x_2, y_2, z_2]$ of system (13).
- This way we obtain a list named *list1* having elements of the form

$$[\alpha, \beta, x_1, y_1, z_1, x_2, y_2, z_2, c_u, b_{max}],$$

b_{max} is defined as

$$b_{max} := \min\{c_u, \lfloor {c_u}^{z_1/x_1} \rfloor, \lfloor {c_u}^{z_1/y_1} \rfloor, \lfloor {c_u}^{z_2/x_2} \rfloor, \lfloor {c_u}^{z_2/y_2} \rfloor\}.$$

Efficient sieve

In order to create *list*2 composed of all possible tuples $[a, b, x_1, y_1, z_1, x_2, y_2, z_2]$ we proceed as follows.

begin
 for each element of *list*1 do
 $T_b := \lceil (b_0 - s_b)/2^\alpha \rceil$
  for $b := T_b \cdot 2^\alpha + s_b$ to b_{max} by 2^α do
  $a_{min} := \max\left\{ a_1, \left\lceil \left(1 + 1/a_1^{x_1}\right)^{-1/x_2} \cdot b^{\frac{y_1 z_2}{x_2 z_1}} \right\rceil \right\}$
  $a_{max} := \min\left\{ c_u, \lfloor {c_u}^{z_1/x_1} \rfloor, \lfloor {c_u}^{z_2/x_2} \rfloor, \left\lfloor (1 + 1/c_2)^{z_2/(x_2 z_1)} \cdot b^{\frac{y_1 z_2}{x_2 z_1}} \right\rfloor \right\}$
  $T_a := \lceil (a_{min} - s_a)/2^\alpha \rceil$
  for $a := T_a \cdot 2^\alpha + s_a$ to a_{max} by 2^α do
   test whether equation $(a^{x_1} + b^{y_1})^{z_2} = (a^{x_2} + b^{y_2})^{z_1}$ holds or not
   put the result $[a, b, x_1, y_1, z_1, x_2, y_2, z_2]$ into the *list*2

Efficient sieve

- As a conclusion, we get that under the assumption $x_1 < x_2$ no solution to equation

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- These contradict the fact that system (13) can have solutions only with $x_1 < x_2$ or $y_1 < y_2$.
- The total computational time was 25 hours on a usual laptop.

Thank you for your attention!