Tatuzawa's theorem for Rankin–Selberg L-functions

(joint work with Jesse Thorner)

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12 June 2026
Online Number Theory Seminar
University of Debrecen

Nonvanishing of Dirichlet L-functions (1 of 2)

Theorem (Hadamard 1896, de la Vallée Poussin 1896 & 1899, Gronwall 1913, Landau 1918, Titchmarsh 1930, McCurley 1984)

If χ is a primitive Dirichlet character, then $L(\sigma + it, \chi)$ has at most one zero (necessarily real and simple) in the region

$$\sigma \geqslant 1 - \frac{1}{10 \log(C(\chi)(|t| + 3))}.$$

If the exceptional zero exists, then $\chi^2 = 1$. Moreover, if χ_1 and χ_2 are two distinct primitive Dirichlet characters, then $L(s, \chi_1)L(s, \chi_2)$ has at most one zero (counted with multiplicity) in the interval

$$\sigma \geqslant 1 - \frac{1}{10 \log(C(\chi_1)C(\chi_2))}.$$

Nonvanishing of Dirichlet L-functions (2 of 2)

Theorem (Siegel 1935, Tatuzawa 1951, Rademacher 1959)

If $\varepsilon \in (0,1)$, then the following holds for all but one primitive quadratic Dirichlet character χ:

$$L(\sigma,\chi) \neq 0, \qquad \sigma \geqslant 1 - \frac{\varepsilon^3}{240} C(\chi)^{-\varepsilon}.$$

Theorem (Walfisz 1936)

If $A > 0$ and $r \pmod{q}$ is a reduced residue class modulo q, then

$$\sum_{\substack{p \leqslant x \\ p \equiv r \pmod{q}}} \log p = \frac{x}{\varphi(q)} + O_A\left(\frac{x}{(\log x)^A}\right).$$

Standard L-functions and Rankin–Selberg L-functions

Cuspidal representations

Let $\mathfrak{F}_n$ be the set of unitary cuspidal automorphic representations of GL_n over a fixed number field F.
Let $\mathfrak{F}_n^* \subset \mathfrak{F}_n$ be the subset of representations in $\mathfrak{F}_n$ whose central character is trivial on the diagonally embedded positive reals.

Each $\pi \in \mathfrak{F}_n$ gives rise to a standard L-function $L(s, \pi)$, which has similar properties as the product of n Hecke L-functions (over F). In fact the product of n Hecke L-functions is the L-function of an isobaric automorphic representation of GL_n over F.

Each $(\pi, \rho) \in \mathfrak{F}_n \times \mathfrak{F}_m$ gives rise to a Rankin–Selberg L-function $L(s, \pi \times \rho)$, which has similar properties as the product of nm Hecke L-functions. Langlands functoriality predicts that $L(s, \pi \times \rho)$ is a product of standard L-functions. Hoffstein–Ramakrishnan (1995) used this hypothesis to prove the non-existence of Landau–Siegel zeros other than those of Hecke L-functions.

Twisting and normalizing cuspidal representations

GL_1-twists

$\mathfrak{F}_1$ is the abelian group of unitary Hecke characters acting on $\mathfrak{F}_n$ as follows. For each $\pi \in \mathfrak{F}_n$ and $\chi \in \mathfrak{F}_1$, we denote by $\pi \otimes \chi \in \mathfrak{F}_n$ the representation $g \mapsto \pi(g)\chi(\det g)$ embedded into the cuspidal subspace of $L^2(\mathrm{GL}_n(F)\backslash\mathrm{GL}_n(\mathbb{A}_F))$ in the usual way.

A special case of this action results in the shifting of the L-function by purely imaginary numbers it ($t \in \mathbb{R}$):

$$L(s + it, \pi) = L(s, \pi \otimes |\cdot|^{it}),$$
$$L(s + it, \pi \times \rho) = L(s, \pi \times (\rho \otimes |\cdot|^{it})).$$

There is a unique decomposition $\pi = \pi^* \otimes |\cdot|^{it_\pi}$ with $\pi^* \in \mathfrak{F}_n^*$ and $t_\pi \in \mathbb{R}$, and similarly for ρ. It follows that

$$L(s, \pi) = L(s + it_\pi, \pi^*),$$
$$L(s, \pi \times \rho) = L(s + it_\pi + it_\rho, \pi^* \times \rho^*).$$

Nonvanishing of Rankin–Selberg L-functions

Establishing zero-free regions for automorphic L-functions has a long history: Jacquet–Shalika (1976), Shahidi (1981), Moreno (1985), **Hoffstein–Lockhart (1994)**, **Goldfeld–Hoffstein–Lieman (1994)**, **Hoffstein–Ramakrishnan (1995)**, Banks (1997), Ramakrishnan–Wang (2003), Iwaniec–Kowalski (2004), Sarnak (2004), Gelbart–Lapid (2006), Goldfeld–Li (2018), Humphries (2019), Jiang–Lü–Thorner–Wang (2023), Luo (2023), Zhang (2023), Wattanawanichkul (2025), Thorner–Zhao (2026).

Theorem (Brumley 2006–2019, Humphries–Thorner 2022)

There exists $c_1 = c_1(n, m, [F:\mathbb{Q}]) > 0$ with the following property. If $(\pi, \rho) \in \mathfrak{F}_n^* \times \mathfrak{F}_m^*$, then $L(\sigma + it, \pi \times \rho)$ has no zero in the region

$$\sigma \geqslant 1 - c_1 (C(\pi)C(\rho))^{-n-m}(|t|+1)^{-nm}.$$

Moreover, if $\pi = \widetilde{\pi}$ or $\rho = \widetilde{\rho}$ or $\rho = \widetilde{\pi}$, then $L(\sigma + it, \pi \times \rho)$ has at most one zero (necessarily real and simple) in the region

$$\sigma \geqslant 1 - c_1 / \log(C(\pi)C(\rho)(|t|+3)).$$

If the exceptional zero exists, then $(\pi, \rho) = (\widetilde{\pi}, \widetilde{\rho})$ or $\rho = \widetilde{\pi}$.

A new zero-free region for $L(s,\pi\times(\tilde{\pi}\otimes\chi))$

Theorem (Harcos–Thorner 2025)

If $(\pi,\chi)\in\mathfrak{F}_n\times\mathfrak{F}_1^*$, *then* $L(\sigma+it,\pi\times(\tilde{\pi}\otimes\chi))$ *has at most one zero (necessarily real and simple) in the region*

$$\sigma\geqslant 1-\frac{1}{903\log(C(\pi)^{2n}C(\chi)^{n^2}(|t|+3)^{n^2[F:\mathbb{Q}]})}.$$

If the exceptional zero exists, then $\pi\otimes\chi^2=\pi$. *Moreover, if* $(\pi,\chi_1,\chi_2)\in\mathfrak{F}_n\times\mathfrak{F}_1^*\times\mathfrak{F}_1^*$ *satisfies* $\pi\otimes\chi_1\neq\pi\otimes\chi_2$, *then* $L(s,\pi\times(\tilde{\pi}\otimes\chi_1))L(s,\pi\times(\tilde{\pi}\otimes\chi_2))$ *has at most one real zero (counted with multiplicity) in the interval*

$$\sigma\geqslant 1-\frac{1}{158\log(C(\pi)^{4n}C(\chi_1)^{n^2}C(\chi_2)^{n^2})}.$$

Remark

If $\pi\otimes\chi^2\neq\pi$, we can replace $C(\chi)^{n^2}(|t|+3)^{n^2[F:\mathbb{Q}]}$ by $C(\chi|\cdot|^{it})^{n^2}$.

The Key Lemma

Let $\Pi = \pi_1 \boxplus \cdots \boxplus \pi_\ell$ be an isobaric sum of unitary cuspidal automorphic representations. Hoffstein–Ramakrishnan (1995) proved that $\log L(s, \Pi \times \widetilde{\Pi})$ has nonnegative Dirichlet coefficients. The same is true of $L(s, \Pi \times \widetilde{\Pi})$ and $-L'(s, \Pi \times \widetilde{\Pi})/L(s, \Pi \times \widetilde{\Pi})$.

Key Lemma (after Goldfeld–Hoffstein–Lieman 1994)

Let r be the order of the pole of $L(s, \Pi \times \widetilde{\Pi})$ at $s = 1$. Let $E(\Pi \times \widetilde{\Pi})$ be $\log C(\Pi \times \widetilde{\Pi})$ plus the sum of reciprocal imaginary parts of the poles of $L(s, \Pi \times \widetilde{\Pi})$ in the upper half-plane, counted with multiplicity. Then $L(\sigma + i\tau, \Pi \times \widetilde{\Pi})$ has at most r zeros (counted with multiplicity) in the region

$$\sigma \geqslant 1 - \frac{1}{7rE(\Pi \times \widetilde{\Pi})} \qquad \text{and} \qquad |\tau| \leqslant \frac{1}{12\sqrt{r}E(\Pi \times \widetilde{\Pi})}.$$

Deducing the theorem from the Key Lemma

1. Assume that $\sigma + it$ is an exceptional zero of $L(s, \pi \times (\widetilde{\pi} \otimes \chi))$. If $\pi \otimes \chi^2 \neq \pi$ or t is "not tiny", we obtain a contradiction by setting $\Pi = \pi \boxplus \pi \otimes \chi|\cdot|^{it} \boxplus \pi \otimes \overline{\chi}|\cdot|^{-it}$ and $\tau = 0$ in the lemma. If $\pi \otimes \chi^2 = \pi$ and t is "tiny" but nonzero, we obtain a contradiction by setting $\Pi = \pi \boxplus \pi \otimes \overline{\chi}$ and $\tau = t$ in the lemma.

2. Assume that $\pi \otimes \chi^2 = \pi$ and $L(s, \pi \times (\widetilde{\pi} \otimes \chi))$ has at least two real exceptional zeros (counted with multiplicity). We obtain a contradiction by setting $\Pi = \pi \boxplus \pi \otimes \overline{\chi}$ and $\tau = 0$ in the lemma.

3. Assume that $L(s, \pi \times (\widetilde{\pi} \otimes \chi_1))L(s, \pi \times (\widetilde{\pi} \otimes \chi_2))$ has at least two real exceptional zeros (counted with multiplicity). We obtain a contradiction by setting $\Pi = \pi \boxplus \pi \otimes \chi_1 \boxplus \pi \otimes \chi_2$ and $\tau = 0$ in the lemma.

A new zero-free region for $L(s,\pi\times(\rho\otimes\chi))$

Theorem (Harcos–Thorner 2025)

Let $(\pi,\rho,\chi)\in\mathfrak{F}_n\times\mathfrak{F}_m\times\mathfrak{F}_1$ and $\varepsilon>0$. There exist an effectively computable constant $c_2=c_2(n,m,[F:\mathbb{Q}],\varepsilon)>0$ and a character $\psi=\psi_{\pi,\rho,\varepsilon}\in\mathfrak{F}_1$ such that if $L(s,\pi\times(\rho\otimes\chi))$ differs from $L(s,\pi\times(\rho\otimes\psi))$, then

$$L(\sigma,\pi\times(\rho\otimes\chi))\neq 0,\qquad \sigma\geqslant 1-c_2(C(\pi)C(\rho)C(\chi))^{-\varepsilon}.$$

Moreover, $L(s,\pi\times(\rho\otimes\psi))$ has at most one real zero (necessarily simple) in the interval $\sigma\geqslant 1-c_2(C(\pi)C(\rho)C(\psi))^{-\varepsilon}$.

Corollary

If $(\pi,\rho)\in\mathfrak{F}_n\times\mathfrak{F}_m$ and $\varepsilon>0$, then $L(\sigma+it,\pi\times\rho)$ has at most one zero (necessarily simple) in the region

$$\sigma\geqslant 1-c_2(C(\pi)C(\rho)D_F(|t|+3)^{[F:\mathbb{Q}]})^{-\varepsilon}.$$

The Key Proposition

The proof relies on the observation that the desired zero-free interval can be established under some auxiliary assumptions.

Key Proposition

Let $(\pi, \rho, \chi) \in \mathfrak{F}_n \times \mathfrak{F}_m \times \mathfrak{F}_1$ *and* $\varepsilon \in (0,1)$. *Put*

$$Q = Q(\pi, \rho, \chi) = (C(\pi)C(\rho))^{2(n+m)} C(\chi)^{(n+m)^2}.$$

Assume that $L(s, \pi \times (\rho \otimes \chi))$ *is entire, and one of the following two conditions holds true:*

1. $L(s, \pi \times \widetilde{\pi})$ *has a zero in the interval* $[1 - \varepsilon/16, 1)$.
2. $L(s, \pi \times \rho)$ *is entire and has a zero in the interval* $[1 - \varepsilon/16, 1)$. *Furthermore, if* $\pi \otimes \chi^* = \pi$ *or* $\rho \otimes \chi^* = \rho$, *then* $|t_\chi| \geqslant Q^{-\varepsilon/64}$.

There exists an effectively computable constant $c_3 = c_3(n, m, [F : \mathbb{Q}], \varepsilon) > 0$ *such that*

$$L(\sigma, \pi \times (\rho \otimes \chi)) \neq 0, \qquad \sigma \geqslant 1 - c_3 Q^{-\varepsilon}.$$

Deducing the theorem from the Key Proposition (1 of 2)

We shall apply the Key Proposition with $(\varepsilon', \rho', \chi')$ such that

$$\varepsilon' = \frac{\varepsilon}{2(n+m)}, \qquad \rho' \otimes \chi' = \rho \otimes \chi.$$

If $L(s, \pi \times \widetilde{\pi})$ has a zero in the interval $[1 - \varepsilon'/16, 1)$, then we are done by the Key Proposition and the standard zero-free region for $L(s, \pi \times \widetilde{\pi})$ established by Humphries–Thorner (2022). Otherwise, $L(s, \pi \times \widetilde{\pi})$ has no exceptional zero, and we can focus on $\chi \in \mathfrak{F}_1$ such that $L(s, \pi \times (\rho \otimes \chi))$ is entire. Let S be the set of such χ.

If, for all $\chi \in S$, the L-function $L(s, \pi \times (\rho \otimes \chi))$ has no zero in $[1 - \varepsilon'/16, 1)$, then we are done. Otherwise, we can fix $\lambda \in S$ with minimal analytic conductor such that $L(s, \pi \times (\rho \otimes \lambda))$ has a zero in $[1 - \varepsilon'/16, 1)$. From now on, we can assume that $C(\chi) \geqslant C(\lambda)$.

Deducing the theorem from the Key Proposition (2 of 2)

We make the change of variables

$$\rho' = \rho \otimes \lambda, \qquad \chi' = \chi\overline{\lambda}.$$

If the twist equivalence condition of the Key Proposition is satisfied for (ρ', χ') in place of (ρ, χ), then we are done. Otherwise,

$$L(s, \pi \times (\rho \otimes \chi)) = L(s + it_\chi - it_\lambda, \pi \times (\rho \otimes \lambda)), \qquad t_\chi \approx t_\lambda.$$

So an exceptional twist $L(s, \pi \times (\rho \otimes \chi))$ and its exceptional real zero σ correspond bijectively to a zero ≈ 1 of $L(s, \pi \times (\rho \otimes \lambda))$:

$$(\chi\text{-twist}, \sigma) \qquad \longleftrightarrow \qquad \sigma + it_\chi - it_\lambda \approx 1.$$

However, applying the Key Lemma for $\Pi = \pi \boxplus \widetilde{\rho} \otimes \overline{\lambda}$ reveals that $L(s, \pi \times (\rho \otimes \lambda))$ has at most one such zero (with multiplicity):

$$L(s, \Pi \times \widetilde{\Pi}) = L(s, \pi \times \widetilde{\pi}) L(s, \rho \times \widetilde{\rho}) L(s, \pi \times (\rho \otimes \lambda)) L(s, \widetilde{\pi} \times (\widetilde{\rho} \otimes \overline{\lambda})).$$

Proof of the Key Proposition (1 of 2)

Generic case: $L(s, \pi \times (\rho \otimes \chi^2))$ is entire in condition ❷.

We use the auxiliary L-function $L(s, \Pi \times \widetilde{\Pi})$, where Π is one of:

1. $\Pi = \pi \boxplus \widetilde{\rho} \otimes \overline{\chi}$
2. $\Pi = \pi \boxplus \pi \otimes \chi \boxplus \widetilde{\rho} \boxplus \widetilde{\rho} \otimes \overline{\chi}$

Hence $L(s, \Pi \times \widetilde{\Pi})$ has nonnegative Dirichlet coefficients, and it has a zero $\beta \in [1 - \varepsilon/16, 1)$ coming from its appropriate factor $L(s, \pi \times \widetilde{\pi})$ or $L(s, \pi \times \rho)$. By an application of Perron's formula and the Residue Theorem, we can show that

$$|L(1, \pi \times (\rho \otimes \chi))| \gg_{n,m,[F:\mathbb{Q}],\varepsilon} (1 - \beta)^4 Q^{-\varepsilon/2}.$$

On the other hand, the Key Lemma actually implies that either $1 - \beta$ exceeds $\varepsilon 2^{-14} Q^{-\varepsilon/64}$, or $L(s, \pi \times (\rho \otimes \chi))$ has no real zero exceeding $1 - \varepsilon 2^{-14} Q^{-\varepsilon/64}$. In both cases we win.

Proof of the Key Proposition (2 of 2)

Non-generic case: $L(s, \pi \times (\rho \otimes \chi^2))$ has a pole in condition ❷.

In this case, $\rho \otimes \chi^2 = \widetilde{\pi} \otimes |\cdot|^{it}$ holds with a unique $t \in \mathbb{R}$.
Consider the Hecke character $\kappa = \overline{\chi}|\cdot|^{it}$, which satisfies

$$\widetilde{\pi} \otimes \kappa = \rho \otimes \chi, \qquad C(\kappa) < (C(\pi)C(\rho)C(\chi)^3)^{[F:\mathbb{Q}]}.$$

The L-functions

$$L(s, \pi \times (\widetilde{\pi} \otimes \kappa)) = L(s, \pi \times (\rho \otimes \chi))$$
$$L(s, \pi \times (\widetilde{\pi} \otimes \kappa^2)) = L(s + it, \pi \times \rho)$$

are entire by assumption. Therefore, we are done by the first discussed new result (applied with κ in the role of χ):

$$L(\sigma, \pi \times (\widetilde{\pi} \otimes \kappa)) \neq 0, \qquad \sigma \geqslant 1 - \frac{1}{903 \log(C(\pi)^{2n} C(\kappa)^{n^2})}.$$