A Moduli Space of Monogenerators

Hanson Smith

University of Connecticut

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Background and Previous Work

Monogenicity/Monogeneity

One of the primary interests of number theory is understanding the roots of monic polynomials in $\mathbb{Z}[x]$. When and how can the roots of one polynomial be expressed by the roots of another polynomial?

Let $K/\mathbb{Q}$ be a number field of degree n with ring of integers $\mathcal{O}_K$. We say *K is monogenic* or $\mathcal{O}_K$ admits a *power integral basis* if $\mathcal{O}_K = \mathbb{Z}[\alpha]$ for some $\alpha \in K$. More explicitly, $\{1, \alpha, \ldots, \alpha^{n-1}\}$ is an $\mathbb{Z}$-basis for the $\mathbb{Z}$-module $\mathcal{O}_K$. In this case we call α a *monogenerator*.

Our First Friends

Take $\mathbb{Q}(\sqrt{d})$ with d square-free. The ring of integers of $\mathbb{Q}(\sqrt{d})$ is $\mathbb{Z}\left[\frac{1+\sqrt{d}}{2}\right]$ if $d \equiv 1 \bmod 4$ and $\mathbb{Z}[\sqrt{d}]$ otherwise. In both cases $\mathbb{Q}(\sqrt{d})$ is monogenic.

Let ζ_n be a primitive n^{th} root of unity and consider the n^{th} cyclotomic field $\mathbb{Q}(\zeta_n)$. It is a bit more difficult than in the quadratic case, but one can show that the ring of integers of $\mathbb{Q}(\zeta_n)$ is $\mathbb{Z}[\zeta_n]$.

The maximal real subfield of the n^{th} cyclotomic field is $\mathbb{Q}(\zeta_n + \zeta_n^{-1})$. These number fields are also monogenic with $\zeta_n + \zeta_n^{-1} = 2\cos(2\pi/n)$ providing a generator.

An Example With Westlund and Wieferich

Proposition
Let p be an odd prime and suppose $x^p + y^p = z^p$ is a non-trivial solution to the Fermat equation. Suppose that $p \nmid xyz$ (the "first case" of Fermat's last theorem). Then, $\sqrt[p]{2}$ is not a monogenerator for $\mathbb{Q}(\sqrt[p]{2})/\mathbb{Q}$.

Let p be a prime and $a \in \mathbb{Z}$ be squarefree. One can show [Westlund, 1910] that $\sqrt[p]{a}$ is a monogenerator if and only if $p^2 \nmid a^p - a$.

[Wieferich, 1909] showed that if $x^p + y^p = z^p$ is a non-trivial solution to the Fermat equation such that $p \nmid xyz$, then $p^2 \mid 2^p - 2$.

"All that glistens is not gold."

Does this always happen? When one is learning (or discovering) algebraic number theory, they might be tempted to think every extension of $\mathbb{Q}$ is monogenic.

Modest doubt is called the beacon of the wise.

- William Shakespeare

Expectation is the root of all heartache.

- William Shakespeare

Dedekind-Kummer Factorization

Theorem (Dedekind building on work of Kummer)

Let $f(x)$ be a monic, irreducible polynomial in $\mathbb{Z}[x]$ with α denoting a root. If $p \in \mathbb{Z}$ is a prime that does not divide $[\mathcal{O}_{\mathbb{Q}(\alpha)} : \mathbb{Z}[\alpha]]$, then the factorization of p in $\mathcal{O}_{\mathbb{Q}(\alpha)}$ mirrors the factorization of $f(x)$ in $\mathbb{F}_p[x]$. That is,

$$f(x) \equiv f_1(x)^{e_1} \cdots f_r(x)^{e_r} \bmod p \quad \textit{and} \quad (p) = \mathfrak{p}_1^{e_1} \cdots \mathfrak{p}_r^{e_r}.$$

For example, consider $\mathbb{Q}(\alpha)$ where α is a root of $x^3 - x^2 - 2x - 8$. Dedekind computed the factorization $(2) = \mathfrak{p}_1\mathfrak{p}_2\mathfrak{p}_3$.

Thus, if this field is monogenic, there is a cubic polynomial $g(x)$ that generates $\mathbb{Q}(\alpha)$ and has **three** distinct linear factors in $\mathbb{F}_2[x]$. In this case we say 2 is a *common index divisor.*

Previous Work

The question of which rings of integers are monogenic was posed to the London Mathematical Society in the 1960's by Helmut Hasse, so the study of monogenicity is sometimes known as *Hasse's problem*.

For an in-depth look at monogenicity with a focus on algorithms for solving index form equations, see Gaál's book [Gaál, 2019]. Evertse and Győry's book [Evertse and Győry, 2017] provides background with a special focus on the relevant Diophantine equations.

For very recent English translations of the original pioneering works, consult [Gouvêa and Webster, 2021a] and [Gouvêa and Webster, 2021b].

Previous Work

The monogenicity of a given extension of $\mathbb{Z}$ is encoded by a Diophantine equation called the *index form equation*. Győry made the initial breakthrough regarding the resolution of index form equations and related equations in the series of papers [Győry, 1973], [Győry, 1974], [Győry, 1976], [Győry, 1978a], and [Győry, 1978b]. For inequivalent monogenic generators one should also consult [Evertse and Győry, 1985], [Bérczes et al., 2013], and the survey

In large part due to the group in Debrecen, there is a vast literature involving relative monogenicity: [Győry, 1980], [Győry, 1981], [Gaál, 2001], [Gaál and Pohst, 2000], [Gaál and Szabó, 2013], [Gaál et al., 2016], [Gaál and Remete, 2019], and [Gaál and Remete, 2019].

Previous Work

Monogenicity has recently been viewed from the perspective of arithmetic statistics: Bhargava, Shankar, and Wang [Bhargava et al., 2016] have shown that the density of monic, irreducible polynomials in $\mathbb{Z}[x]$ such that a root is a monogenerator is $\frac{6}{\pi^2} = \zeta(2)^{-1} \approx 60.79\%$.

They also show the density of monic integer polynomials with square-free discriminants (a sufficient condition for a root to be a monogenerator) is

$$\prod_p \left(1 - \frac{1}{p} + \frac{(p-1)^2}{p^2(p+1)}\right) \approx 35.82\%.$$

Following the completion of the work I will be speaking about, we were made aware of several bodies of related work on monogenicity in geometric contexts, such as Stein manifolds [Duchamp and Hain, 1984], [Stout, 1972], [Stutz, 1974] and Riemann surfaces [Prill, 1980]. The question of the existence of monogenerators, referred to as "primitive elements" was raised by Alling for Riemann surfaces [Alling, 1964] and then for Stein manifolds by Röhrl [Röhrl, 1965].

Construction

The Classical Index Form

Let K be a number field. If $f \in \mathbb{Z}[x]$ is an irreducible monic polynomial with a root α such that $K = \mathbb{Q}(\alpha)$, then

$$\mathrm{Disc}(f) = \mathrm{Disc}(K)\left[\mathcal{O}_K : \mathbb{Z}[\alpha]\right]^2.$$

Here $[\mathcal{O}_K : \mathbb{Z}[\alpha]]$ is the index of $\mathbb{Z}[\alpha]$ inside $\mathcal{O}_K$. We see that this index is 1 if and only if α is a monogenerator. Thus we can define "the" *index form* of $\mathcal{O}_K/\mathbb{Z}$ by

$$\mathrm{Index}_{\mathcal{O}_K/\mathbb{Z}}(\alpha) = \sqrt{\frac{\mathrm{Disc}(f)}{\mathrm{Disc}(K)}}.$$

The Classical Index Form

More explicitly, let $\{\beta_1, \ldots, \beta_n\}$ be an integral basis for $\mathcal{O}_K$ and let $\{\sigma_1, \ldots, \sigma_n\}$ be the set of embeddings of $K \hookrightarrow \mathbb{C}$. We have

$$\mathrm{Disc}(K) = \det \begin{bmatrix} \sigma_1(\beta_1) & \sigma_1(\beta_2) & \cdots & \sigma_1(\beta_n) \\ \sigma_2(\beta_1) & \ddots & & \sigma_2(\beta_n) \\ \vdots & & \ddots & \vdots \\ \sigma_n(\beta_1) & \cdots & \cdots & \sigma_n(\beta_n) \end{bmatrix}^2 .$$

If $\underset{\beta\to\alpha}{M}$ is the appropriate change of (K-)basis matrix from $\{\beta_1, \ldots, \beta_n\}$ to $\{1, \alpha, \alpha^2, \ldots, \alpha^{n-1}\}$, then

$$\underset{\beta\to\alpha}{M} \begin{bmatrix} \sigma_1(\beta_1) & \sigma_1(\beta_2) & \cdots & \sigma_1(\beta_n) \\ \sigma_2(\beta_1) & \ddots & & \sigma_2(\beta_n) \\ \vdots & & \ddots & \vdots \\ \sigma_n(\beta_1) & \cdots & \cdots & \sigma_n(\beta_n) \end{bmatrix} = \begin{bmatrix} \sigma_1(1) & \sigma_1(\alpha) & \cdots & \sigma_1(\alpha^{n-1}) \\ \sigma_2(1) & \ddots & & \sigma_2(\alpha^{n-1}) \\ \vdots & & \ddots & \vdots \\ \sigma_n(1) & \cdots & \cdots & \sigma_n(\alpha^{n-1}) \end{bmatrix}.$$

The Classical Index Form

Taking determinants and squaring, we have

$$\det \begin{pmatrix} M \\ \beta\to\alpha \end{pmatrix}^2 \mathrm{Disc}(K) = \mathrm{Disc}\left(1, \alpha, \ldots, \alpha^{n-1}\right) = \mathrm{Disc}(f).$$

Therefore, $\det \begin{pmatrix} M \\ \beta\to\alpha \end{pmatrix}^2 = \mathrm{Index}_{\mathcal{O}_K/\mathbb{Z}}(\alpha)^2$ is an equality in $\mathbb{Z}$, so

$$\det \begin{pmatrix} M \\ \beta\to\alpha \end{pmatrix} = \pm\mathrm{Index}_{\mathcal{O}_K/\mathbb{Z}}(\alpha).$$

This is really putting the cart before the horse. See Theorem 5.19 of https://kconrad.math.uconn.edu/blurbs/linmultialg/modulesoverPID.pdf for a clear elementary proof.

Starting to Think More Geometrically

Definition
$\alpha \in \mathcal{O}_K$ is a *monogenerator* for $\mathcal{O}_K/\mathbb{Z}$ if there is a commutative diagram

$$\begin{array}{ccc} & & \mathbb{Z}[t] \\ & \swarrow & \uparrow \\ \mathcal{O}_K & \longleftarrow & \mathbb{Z} \end{array}$$

where the diagonal ring homomorphism sending $t \mapsto \alpha$ is surjective.

Schemes

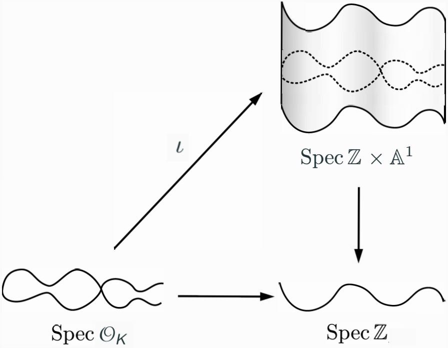
We continue our translation by applying Spec to get some schemes.

The essential points on schemes:

- If A is a commutative ring, $\mathrm{Spec} A$ is a geometric object (an "affine scheme") whose points are the prime ideals of A.
- A scheme in general is a geometric object built by taking a union of affine schemes.
- A morphism $\pi : \mathrm{Spec} A \to \mathrm{Spec} B$ is the same as a ring homomorphism $\pi^\sharp : B \to A$. We say that π is a *closed immersion* if $\pi^\sharp$ is surjective.

A Geometric Definition

Definition
A monogenerator for $\mathcal{O}_K/\mathbb{Z}$ is a closed immersion $\iota : \mathrm{Spec}\mathcal{O}_K \to \mathrm{Spec}\mathbb{Z}[t]$ so that



commutes.

Monogenicity of Schemes

Definition
Let $\pi : S' \to S$ be a finite, locally free morphism of Noetherian schemes of constant rank. We say that a monogenerator for $S' \to S$ is a diagram

$$\begin{array}{ccc} & & \mathbb{A}^1_S \\ & \overset{\iota}{\nearrow} & \downarrow \\ S' & \longrightarrow & S \end{array}$$

where ι is a closed immersion.

The Scheme of Monogenerators

Definition
Let $\pi : S' \to S$ be as in the last slide. We define the *scheme of monogenic generators* $\mathcal{M}_{S'/S}$ to be the S-scheme with the property that for any S-scheme T,

$$\mathrm{Hom}_{\mathrm{Sch}/S}(T, \mathcal{M}_{S'/S}) = \left\{ \begin{array}{ccc} & & \mathbb{A}^1_T \\ & \nearrow^{\iota} & \downarrow \\ S' \times_S T & \longrightarrow & T \end{array} \right\}.$$

Theorem
$\mathcal{M}_{S'/S}$ exists. Moreover, when S is affine, then so is $\mathcal{M}_{S'/S}$.

The Scheme of Monogenerators

A special case

$$\mathrm{Hom}_{\mathrm{Sch}/S}(\mathrm{Spec}B, \mathcal{M}_{\mathcal{O}_K/\mathbb{Z}}) = \begin{Bmatrix} \text{monogenic generators for} \\ B \otimes_{\mathbb{Z}} \mathcal{O}_K \text{ over } B \end{Bmatrix}$$

Another special case

$$\mathrm{Hom}_{\mathrm{Sch}/S}(\mathrm{Spec}\mathbb{Z}, \mathcal{M}_{\mathcal{O}_K/\mathbb{Z}}) = \begin{Bmatrix} \text{monogenic generators of } \mathcal{O}_K \\ \text{over } \mathbb{Z}. \end{Bmatrix}$$

Remark

We may also replace the scheme $\mathbb{A}^1_T$ in the definition with $\mathbb{A}^k_T$ to study generating k-tuples of elements.

Examples

Dedekind's Non-Monogenic Cubic

Let α denote a root of the polynomial $x^3 - x^2 - 2x - 8$ and consider the field extension $K = \mathbb{Q}(\alpha)$ over $\mathbb{Q}$. Two generators are necessary to generate $\mathcal{O}_K/\mathbb{Z}$. We take $\mathcal{B} = \{1, \frac{\alpha+\alpha^2}{2}, \alpha^2\}$ as our $\mathbb{Z}$-basis for $\mathcal{O}_K$. We consider the map $\mathbb{Z}[t] \to \mathcal{O}_K$ given by sending t to a generic element $a + b\frac{\alpha+\alpha^2}{2} + c\alpha^2$.

$$\begin{array}{ccc} \mathcal{O}_K[a,b,c] & \xleftarrow[a+b\frac{\alpha+\alpha^2}{2}+c\alpha^2 \longleftarrow x]{} & \mathbb{Z}[a,b,c][x]. \\ & \nwarrow \quad \nearrow & \\ & \mathbb{Z}[a,b,c] & \end{array}$$

Dedekind's Non-Monogenic Cubic

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$$\begin{bmatrix} 1 & a & a^2 + 6b^2 + 16bc + 8c^2 \\ 0 & b & 2ab + 7b^2 + 24bc + 20c^2 \\ 0 & c & -2b^2 + 2ac - 8bc - 7c^2 \end{bmatrix}.$$

Taking the determinant, the index form associated to this basis is

$$-2b^3 - 15b^2c - 31bc^2 - 20c^3.$$

Reducing modulo 2, we find that $b^2c + bc^2 = 1$ has no solutions in $\mathbb{Z}/2\mathbb{Z}$.

An Inseparable Extension of Function Fields

We investigate the analog of the integers in the function field extension $\mathbb{F}_3(\alpha)[\beta]/(\beta^3 - \alpha)$ over $\mathbb{F}_3(\alpha)$. The base ring is $\mathbb{F}_3[\alpha]$ and the extension ring is $\mathbb{F}_3[\alpha][x]/(x^3 - \alpha) = \mathbb{F}_3[\beta]$, where $\beta^3 = \alpha$.

$$\mathbb{F}_3[a,b,c][\beta] \xleftarrow[a+b\beta+c\beta^2 \mapsfrom x]{} \mathbb{F}_3[\alpha][a,b,c][x].$$

$$\mathbb{F}_3[a,b,c][\beta] \leftarrow \mathbb{F}_3[\alpha][a,b,c] \rightarrow \mathbb{F}_3[\alpha][a,b,c][x]$$

$$1 \mapsfrom 1$$

$$a + b\beta + c\beta^2 \mapsfrom x$$

$$(a + b\beta + c\beta^2)^2 \mapsfrom x^2$$

corresponds to the change of basis matrix

$$\begin{bmatrix} 1 & a & a^2 + 2bc\alpha \\ 0 & b & c^2\alpha + 2ab \\ 0 & c & b^2 + 2ac \end{bmatrix}.$$

An Inseparable Extension of Function Fields

From the previous slide, the base ring is $\mathbb{F}_3[\alpha]$ and the extension ring is $\mathbb{F}_3[\alpha][x]/(x^3 - \alpha) = \mathbb{F}_3[\beta]$, where $\beta^3 = \alpha$.

$$\det \begin{bmatrix} 1 & a & a^2 + 2bc\alpha \\ 0 & b & c^2\alpha + 2ab \\ 0 & c & b^2 + 2ac \end{bmatrix} = b^3 - c^3\alpha \in \mathbb{F}_3[\alpha][a, b, c].$$

This is not geometrically reduced since it factors as $(b - c\beta)^3$. To find the monogenerators of this extension, we set this expression equal to the units of $\mathbb{F}_3[\alpha]$. Since $(\mathbb{F}_3[\alpha])^* = \pm 1$, the only solutions are $b = \pm 1$, $c = 0$. Thus

$$\mathcal{M}_{1,\mathbb{F}_3[\beta]/\mathbb{F}_3[\alpha]}(\mathbb{F}_3[\alpha]) = \{a \pm \beta : a \in \mathbb{F}_3[\alpha]\}.$$

We can see that, much like number rings, monogenicity imposes a stronger restriction here than it does for the extension of fraction fields.

So What's the Point?

This translation makes immediately available the machinery and organization provided by scheme theory:

- Monogenerators for $\mathcal{O}_L$ over $\mathcal{O}_K$ are in bijection with $\mathcal{O}_K$ points of $\mathcal{M}_{\mathcal{O}_L/\mathcal{O}_K}$.
- $\mathcal{M}_{S'/S}$ makes good sense for very many $S' \to S$ in a single framework. For example S' might be the spectrum of an order, or $S' \to S$ might be a covering of one curve by another.
- $\mathcal{M}_{\mathcal{O}_L/\mathcal{O}_K}$ is a sheaf with respect to the fpqc, fppf, étale, and Zariski topologies, so searching for monogenerators can be done locally.
- We can consider twists of $\mathcal{M}_{\mathcal{O}_L/\mathcal{O}_K}$ by any groups that act on it and interpret the meaning of their points.

Twisted Monogenicity

Natural Actions on $\mathcal{M}_{S'/S}$

Since

$$\mathrm{Hom}_{\mathrm{Sch}/S}(T, \mathcal{M}_{S'/S}) = \left\{ \begin{array}{ccc} & & \mathbb{A}^1_T \\ & \overset{\iota}{\nearrow} & \downarrow \\ S' \times_S T & \longrightarrow & T \end{array} \right\},$$

we can act by $\mathrm{Aut}(\mathbb{A}^1_S/S)$ or $\mathrm{Aut}(S'/S)$. For $\mathcal{O}_L/\mathcal{O}_K$, the first action translates to

$$\alpha \mapsto m\alpha + b$$

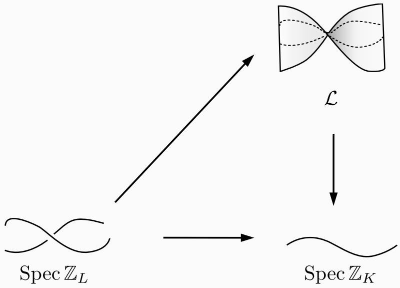
where $m \in \mathcal{O}_K^*, b \in \mathcal{O}_K$, and the second action translates to

$$\alpha \mapsto \sigma(\alpha)$$

where $\sigma \in \mathrm{Gal}(L/K)$.

Twisted Monogenicity: Geometric Definition

Definition
We say L/K is *twisted monogenic* if $\mathrm{Spec}\mathcal{O}_L$ admits a closed immersion into a line bundle $\mathscr{L}$ over $\mathrm{Spec}\mathcal{O}_K$.



An Algebraic Definition

Definition
We say that L/K is *twisted monogenic* if there is

1. a finite set $\{f_1, \ldots, f_m\}$ of relatively prime elements of $\mathcal{O}_K$
2. an element $\alpha_i \in \mathcal{O}_K[f_i^{-1}]$ for each $i = 1, \ldots, m$

so that

1. $\mathcal{O}_L[f_i^{-1}] = \mathcal{O}_K[f_i^{-1}, \alpha_i]$ for each $i = 1, \ldots, m$
2. for each $i, j \in \{1, \ldots, m\}$ there is $a_{ij} \in (\mathcal{O}_K[f_i^{-1}, f_j^{-1}])^*$ and $b_{ij} \in \mathcal{O}_K[f_i^{-1}, f_j^{-1}]$ so that

$$\alpha_i = a_{ij}\alpha_j + b_{ij}.$$

An Example

Example
Let $K = \mathbb{Q}(\sqrt[n]{pq})$ where p, q are distinct primes relatively prime to n. Suppose further that $(n) = \prod_i \mathfrak{p}_i^{e_i}$ with $\mathfrak{p}_i = (\theta_i)$ principal. For example, $n = 3, p = 5$, and $q = 23$ work.

Then the integers of $L = K\left(\sqrt[n]{q\prod_i \theta_i}\right)$ are twisted monogenic over $\mathcal{O}_K$:

- Choose p and q as our relatively prime elements.
- Choose $\alpha_p = \sqrt[n]{p^{n-1}\prod_i \theta_i}$ and $\alpha_q = \sqrt[n]{q\prod_i \theta_i}$.

This works because:

- α_p is a monogenerator for $\mathcal{O}_L[p^{-1}]/\mathcal{O}_K[p^{-1}]$ and α_q is a monogenerator for $\mathcal{O}_L[q^{-1}]/\mathcal{O}_K[q^{-1}]$ by a check with Dedekind's criterion for relative extensions;
- $\alpha_q = \frac{\sqrt[n]{pq}}{p} \cdot \alpha_p$.

An Example

Example continued
When $n = 3, p = 5, q = 23$, a check with Sage shows that $\mathcal{O}_L/\mathcal{O}_K$ is twisted monogenic but *not* monogenic.

Some of these *are* monogenic though!

Twisted Monogenicity and the Class Group

Theorem
If L/K is twisted monogenic and K has trivial class group, then L/K is monogenic.

Theorem
K has trivial class group if and only if every twisted monogenic extension of K is monogenic.

Theorem
If L/K is a twisted monogenic extension of degree n with line bundle $\mathscr{L}$, then the Steinitz class of $\mathcal{O}_L$ over $\mathcal{O}_K$ is the $\frac{n(n-1)}{2}$-th power of the class of $\mathscr{L}$.

Thank You!

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