Associated r-Dowling numbers and some relatives

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Bell numbers

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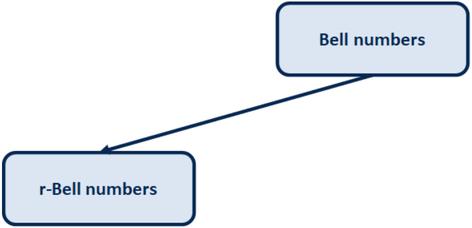
B_n: number of partitions of $\{1, \ldots, n\}$

r-Bell numbers



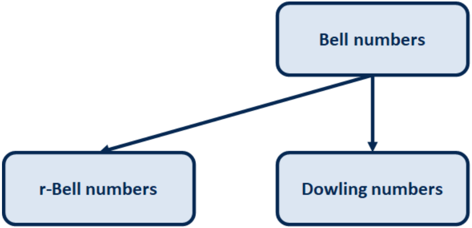
L. Carlitz (1980), I. Mező (2011)

r-partition: a partition of $\{1, \dots, n+r\}$ where $1, \dots, r$ belong to distinct blocks

$B_{n,r}$: number of r-partitions of $\{1, \dots, n+r\}$

$B_{n,0} = B_n$ and $B_{n,1} = B_{n+1}$

Dowling numbers



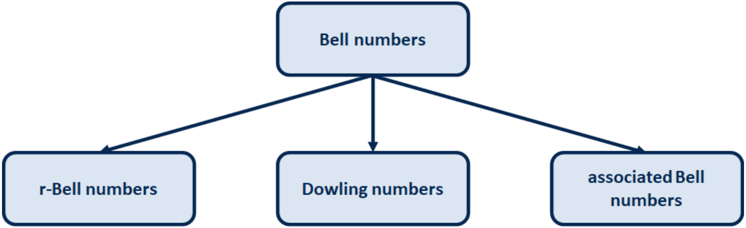
Dowling numbers

M. Benoumhani (1996)

$D_{n,m}$: defined using Whitney numbers in connection with finite groups of order m

$D_{n,1} = B_{n+1}$

s-associated Bell numbers



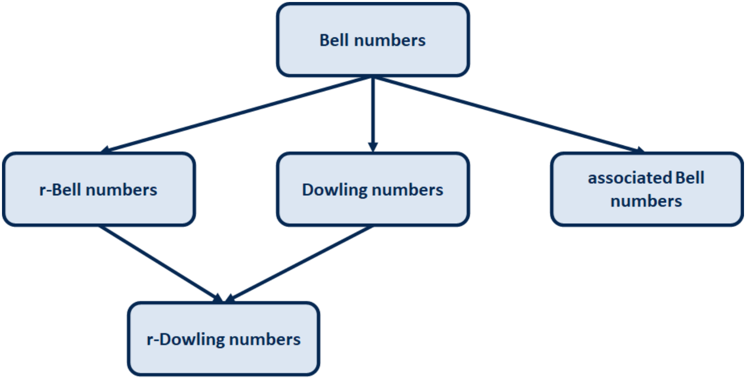
s-associated Bell numbers

E. A. Enneking and J. C. Ahuja (1976), F. T. Howard (1977, 1980), V. H. Moll, J. L. Ramírez and D. Villamizar (2018), M. Bóna and I. Mező (2016)

$B_n^{\geq s}$: number of those partitions of $\{1, \ldots, n\}$, where each block contains at least s elements

$B_n^{\geq 1} = B_n$

r-Dowling numbers



G.-S. Cheon and J.-H. Jung (2012), R. B. Corcino, C. B. Corcino and R. Aldema (2006), E. Gyimesi and G. Nyul (2019)

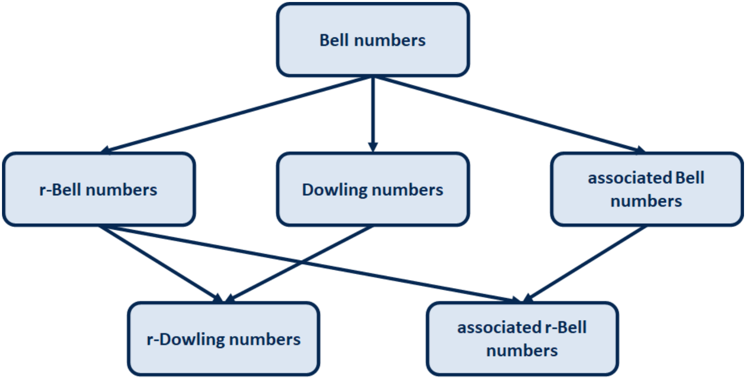
Whitney coloured r-partition with m colours: an r-partition where

- the smallest elements of the blocks are not coloured,
- elements in distinguished blocks are not coloured,
- the remaining elements are coloured with m colours.

$D_{n,m,r}$: number of Whitney coloured r-partitions of $\{1, \ldots, n+r\}$ with m colours

$D_{n,1,r} = B_{n,r}$ and $D_{n,m,1} = D_{n,m}$

s-associated r-Bell numbers

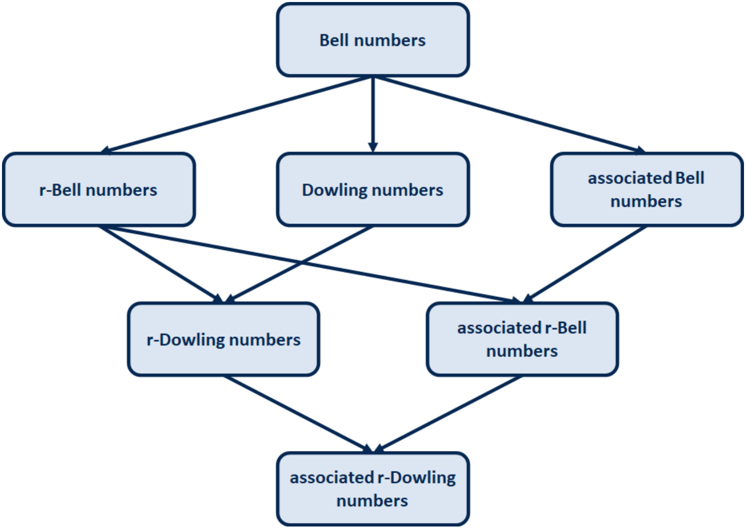


F. T. Howard (1984)

$B_{n,r}^{\geq s}$: number of those r-partitions of $\{1, \ldots, n+r\}$, where each non-distinguished block contains at least s elements

$B_{n,r}^{\geq 1} = B_{n,r}$ and $B_{n,0}^{\geq s} = B_n^{\geq s}$

s-associated r-Dowling numbers



s-associated r-Dowling numbers

Denote by $D_{n,m,r}^{\geq s}$ the total number of Whitney coloured r-partitions of $\{1, \ldots, n+r\}$ with m colours, where each non-distinguished block contains at least s elements.

$D_{n,m,r}^{\geq 1} = D_{n,m,r}$ and $D_{n,1,r}^{\geq s} = B_{n,r}^{\geq s}$

r-permutation: a permutation of $\{1, \dots, n+r\}$ where $1, \dots, r$ belong to distinct cycles

Whitney coloured r-permutation with m colours: an r-permutation where

- the smallest elements of the cycles are not coloured,
- an element in a distinguished cycle is not coloured if there are no smaller numbers on the arc from the distinguished element to this element,
- the remaining elements are coloured with m colours.

The permutational variants

- $A_n = n!$ (number of permutations of $\{1, \ldots, n\}$)
- $A_{n,r} = (r+1)^{\overline{n}}$ (number of r-permutations of $\{1, \ldots, n+r\}$)
- $DA_{n,m} = (2|m)^{\overline{n}}$
- $A_n^{\geq s}$: number of permutations of $\{1, \ldots, n\}$, where each cycle has length at least s
- $DA_{n,m,r} = (r+1|m)^{\overline{n}}$ (number of Whitney coloured r-permutations of $\{1, \ldots, n+r\}$)
- $A_{n,r}^{\geq s}$: number of those r-permutations of $\{1, \ldots, n+r\}$, where each non-distinguished cycle has length at least s

s-associated r-Dowling factorials

Denote by $DA_{n,m,r}^{\geq s}$ the total number of Whitney coloured r-permutations of $\{1, \ldots, n+r\}$ with m colours, where each non-distinguished cycle contains at least s elements.

Partitions into ordered blocks

Whitney–Lah coloured r-partition with m colours: an r-partition into ordered blocks where

- the smallest elements of the ordered blocks are not coloured,
- an element in a distinguished ordered block is not coloured if there are no smaller numbers between the distinguished element and this element,
- the remaining elements are coloured with m colours.

Partitions into ordered blocks

- L_n: number of partitions of $\{1, \ldots, n\}$ into ordered blocks
- $L_{n,r}$: number of r-partitions of $\{1, \ldots, n + r\}$ into ordered blocks
- $DL_{n,m}$
- $L_n^{\geq s}$: number of those partitions of $\{1, \ldots, n\}$ into ordered blocks, where each ordered block contains at least s elements
- $DL_{n,m,r}$: number of Whitney–Lah coloured r-partitions of the set $\{1, \ldots, n + r\}$ with m colours
- $L_{n,r}^{\geq s}$: number of those r-partitions of $\{1, \ldots, n + r\}$ into ordered blocks, where each non-distinguished ordered block contains at least s elements

s-associated r-Dowling–Lah numbers

Denote by $DL_{n,m,r}^{\geq s}$ the total number of Whitney–Lah coloured r-partitions of $\{1, \ldots, n + r\}$ with m colours, where each non-distinguished ordered block contains at least s elements.

Theorem

Let $f_1, f_2, g\colon \mathbb{N}_0 \to \mathbb{K}$ be functions such that $f_2(0) = 0$ and $g(0) = 1$. Denote their exponential generating functions by $F_1(x), F_2(x)$ and $G(x)$, respectively. Define the function $h\colon \mathbb{N}_0 \to \mathbb{K}$ as follows: $h(0) = 1$, and for $n \geq 1$ let

$$h(n) = \sum f_1(|Y_1|) \cdots f_1(|Y_r|)\, f_2(|Z_1|) \cdots f_2(|Z_k|)\, g(k),$$

where the sum is taken for all r-partitions $\{Y_1 \cup \{1\}, \ldots, Y_r \cup \{r\}, Z_1, \ldots, Z_k\}$ of $\{1, \ldots, n+r\}$. Then the exponential generating function of h is

$$H(x) = \left(F_1(x)\right)^r G\left(F_2(x)\right).$$

Exponential generating functions

Theorem

If $r \geq 0$ and $s, m \geq 1$, then

$$\sum_{n=0}^{\infty} \frac{D_{n,m,r}^{\geq s}}{n!} x^n = \exp\left(rx + \frac{\exp(mx) - 1}{m}\right) \exp\left(-\frac{1}{m}\sum_{j=1}^{s-1}\frac{1}{j!}(mx)^j\right),$$

$$\sum_{n=0}^{\infty} \frac{DA_{n,m,r}^{\geq s}}{n!} x^n = (1 - mx)^{-\frac{r+1}{m}} \exp\left(-\frac{1}{m}\sum_{j=1}^{s-1}\frac{1}{j}(mx)^j\right),$$

$$\sum_{n=0}^{\infty} \frac{DL_{n,m,r}^{\geq s}}{n!} x^n$$

$$= (1 - mx)^{-\frac{2r}{m}} \exp\left(\frac{1}{m}\left(\frac{1}{1-mx} - 1\right)\right) \exp\left(-\frac{1}{m}\sum_{j=1}^{s-1}(mx)^j\right).$$

Exponential generating functions

$$\sum_{n=0}^{\infty} \frac{D_{n,m,r}^{\geq s}}{n!} x^n = \exp\left(rx + \frac{\exp(mx) - 1}{m}\right) \exp\left(-\frac{1}{m} \sum_{j=1}^{s-1} \frac{1}{j!}(mx)^j\right)$$

Proof

If

$$f_1(n) = 1, \quad f_2(n) = \begin{cases} 0 & \text{if } n \leq s-1 \\ m^{n-1} & \text{if } n \geq s \end{cases}, \quad g(n) = 1,$$

then $h(n) = D_{n,m,r}^{\geq s}$. For these sequences, we have

$$F_1(x) = \exp(x), F_2(x) = \frac{1}{m}\left(\exp(mx) - \sum_{j=0}^{s-1} \frac{1}{j!}(mx)^j\right), G(x) = \exp(x).$$

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Recurrences I.

Theorem

If $r \geq 0$, $s, m \geq 1$ and $n \geq s-1$, then

$$D^{\geq s}_{n+1,m,r} = rD^{\geq s}_{n,m,r} + \sum_{j=0}^{n-s+1} \binom{n}{j} D^{\geq s}_{j,m,r} m^{n-j},$$

$$DA^{\geq s}_{n+1,m,r} = r\sum_{j=0}^{n} \binom{n}{j} DA^{\geq s}_{j,m,r} m^{n-j} (n-j)! + \sum_{j=0}^{n-s+1} \binom{n}{j} DA^{\geq s}_{j,m,r} m^{n-j} (n-j)!,$$

$$DL^{\geq s}_{n+1,m,r} = 2r\sum_{j=0}^{n} \binom{n}{j} DL^{\geq s}_{j,m,r} m^{n-j} (n-j)! + \sum_{j=0}^{n-s+1} \binom{n}{j} DL^{\geq s}_{j,m,r} m^{n-j} (n-j+1)!.$$

Theorem

If $r \geq 0$, $s, m \geq 1$ and $n \geq s-1$, then

$$DA^{\geq s}_{n+1,m,r} = (mn+r)\, DA^{\geq s}_{n,m,r} + (mn|m)^{\underline{s-1}}\, DA^{\geq s}_{n-s+1,m,r}.$$

If $r \geq 0$, $s, m \geq 1$ and $n \geq s$, then

$$\begin{aligned} DL^{\geq s}_{n+1,m,r} &= (2mn+2r)\, DL^{\geq s}_{n,m,r} + s\,(mn|m)^{\underline{s-1}}\, DL^{\geq s}_{n-s+1,m,r} \\ &\quad - mn\,(mn-m+2r)\, DL^{\geq s}_{n-1,m,r} - (s-1)\,(mn|m)^{\underline{s}}\, DL^{\geq s}_{n-s,m,r}. \end{aligned}$$

Theorem

If $n \geq 0$, $r \geq r' \geq 0$ and $s, m \geq 1$, then

$$D_{n,m,r}^{\geq s} = \sum_{j=0}^{n} \binom{n}{j} D_{j,m,r'}^{\geq s} \left(r - r'\right)^{n-j},$$

$$DA_{n,m,r}^{\geq s} = \sum_{j=0}^{n} \binom{n}{j} DA_{j,m,r'}^{\geq s} \left(r - r'|m\right)^{\overline{n-j}},$$

$$DL_{n,m,r}^{\geq s} = \sum_{j=0}^{n} \binom{n}{j} DL_{j,m,r'}^{\geq s} \left(2r - 2r'|m\right)^{\overline{n-j}}.$$

Theorem

If $n, r \geq 0$ and $s, m \geq 1$, then

$$D^{\geq s}_{n,m,r} = e^{-\frac{1}{m}} \sum_{k=0}^{\infty} \frac{1}{m^k k!} \sum_{*} \frac{n!}{l!} (mk+r)^l \prod_{j=1}^{s-1} \frac{1}{i_j!} \left(-\frac{m^{j-1}}{j!} \right)^{i_j},$$

$$DA^{\geq s}_{n,m,r} = \sum_{*} \frac{n!}{l!} (r+1|m)^{\bar{l}} \prod_{j=1}^{s-1} \frac{1}{i_j!} \left(-\frac{m^{j-1}}{j} \right)^{i_j},$$

$$DL^{\geq s}_{n,m,r} = e^{-\frac{1}{m}} \sum_{k=0}^{\infty} \frac{1}{m^k k!} \sum_{*} \frac{n!}{l!} (mk+2r|m)^{\bar{l}} \prod_{j=1}^{s-1} \frac{1}{i_j!} \left(-m^{j-1} \right)^{i_j},$$

where the sums indicated with a star symbol are taken over all s-tuples $(i_1, i_2, \ldots, i_{s-1}, l)$ of nonnegative integers satisfying $i_1 + 2i_2 + \cdots + (s-1)\, i_{s-1} + l = n$.

2-associated r-Dowling numbers and $(r-1)$-Dowling numbers

Corollary

If $n \geq 0$ and $r, m \geq 1$, then $D^{\geq 2}_{n,m,r} = D_{n,m,r-1}$.

Thank you for your attention!