Integers as sums of three cubes

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Dedicated to Richard Guy

Historical Background

Are all numbers which are not of the form $9n \pm 4$ the sum of *three* cubes? From the list of unknowns given in the second edition, Andrew Bremner has deleted 75 and 600; Conn & Vaserstein 84; Richard Lukes 110, 435 and 478; Kenyi Koyama 444, 501, 618, 912 and 969. Don Reble quoted $30 = 2220422932^3 - 283059965^3 - 2218888517^3$ and the website
`http://www.asahi-net.or.jp/~KC2H-MSM/mathland/math04/cube01.htm`
which implies that only 25 remain:

33	42	52	74	114	156	165	318	366	390	420	564	579
627	633	732	758	789	795	894	906	921	933	948	975	

Noam Elkies sent 462 and many smaller representations of earlier unknowns, and he wrote

> . . . is the representation $12 = 9730705^3 - 9019406^3 - 5725013^3$ known? Miller & Woolett found only $12 = 10^3 + 7^3 - 11^3$ and asked about the existence of further solutions, of which the above is the first. Likewise for $2 = 1214928^3 + 3480205^3 - 3528875^3$, the first instance not accounted for by the identity $2 = (6t^3+1)^3 - (6t^3-1)^3 - (6t^2)^3$.

The equation $3 = x^3 + y^3 + z^3$ has the solutions $(1,1,1)$ and $(4,4,-5)$. Are there any others?

FIGURE: How Richard Guy controls people's lives
(p. 232 from UPINT.v3-2004)

BEATS THE WORLD AT MATHEMATICS

LEWIS J. MORDEL

Lewis J. Mordel, High School Graduate, Wins Scholarship in Cambridge Over Competitors from Many Countries.

Lewis J. Mordel, a graduate of the Central High School, brought additional honors to his alma mater yesterday, when he was awarded a three-year scholarship in mathematics by St. John's College, Cambridge, England.

Mordel went to Cambridge with nothing but his High School training and competed against graduates of schools and colleges in every part of the world. The examinations were open to all competitors, but for the first time a High School graduate was entered against college men. His entry created laughter instead of serious consideration, but at the conclusion of the examinations, which lasted four days, he stood No. 1 of 250 applicants, with an average of a trifle below 100.

At the Central High School Mordel's ability along mathematical lines was regarded by the members of the faculty as phenomenal. In his Sophomore year he had completed the mathematical course provided for the four-year course and during his last two years in the school he took up the higher mathematics.

To support himself he devoted seven hours of every day to coaching his fellow-students, and on one occasion stood at a blackboard for forty-eight hours in an endeavor to pull a student through an examination. And the examination was passed. At the end of his Senior year he devoted all his time to coaching, having no examinations to take, and in this manner earned enough money to take him to England.

Mordel's present aim is to cover his three years' work sufficiently well to entitle him to a fellowship for four additional years.

I do not know anything about the integer solutions of

$$X^3+Y^3+Z^3=3$$

beyond the existence of the four sets (1, 1, 1), (4, 4, —5) etc.; and it must be very difficult indeed to find out anything about any other solutions*.

FIGURE: Mordell's original question, *J. London Math. Soc.* 1953

J.C.P. Miller (Cambridge) and
M.F.C. Woollett (Royal Aeronautical Establishment) **1953-54**

Mordell suggested a computer search for solutions ($k = 3$).

Miller and Woollett searched for integer solutions to

$$x^3 + y^3 + z^3 = k,$$

with $0 \leq k \leq 100, \quad |z| \leq |y| \leq |x| \leq 3164$.

No sol'ns for $k \in \{24, 30, 33, 39, 42, 52, 74, 75, 80, 84, 87, 96\}$.

All sol'ns found for $k = 2$ satisfy
$(x, y, z) = (6t^3 + 1, -6t^3 + 1, -6t^2)$.

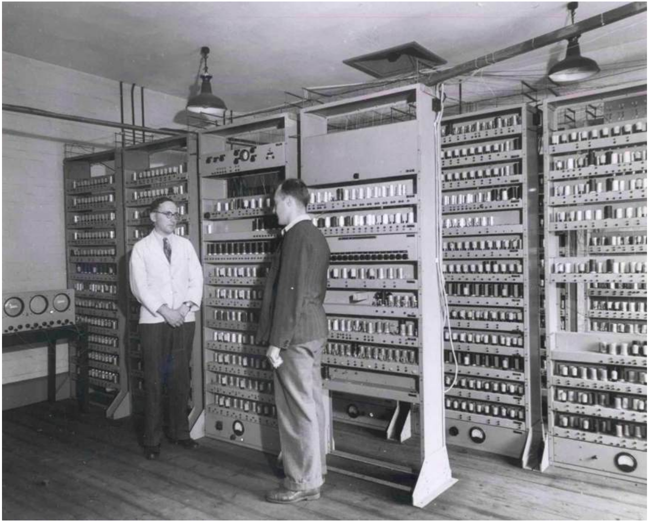

FIGURE: The Electronic Delay Storage Automatic Calculator, 1953-54

V.L. Gardiner, R.B. Lazarus, P.R. Stein (1961-63):
extended the Miller-Woollett search for $|x|$ up to 65536.

· Suggested to Stein (Los Alamos) by **Chowla** (UC-Boulder) (for $k = 3$) in 1961.

· Search was performed in 1963 on the IBM 7030 STRETCH computer at the Los Alamos Research Lab.

· For the unsolved $k \leq 100$, solutions found only for $k = 87$ and $k = 96$.

Historical Background

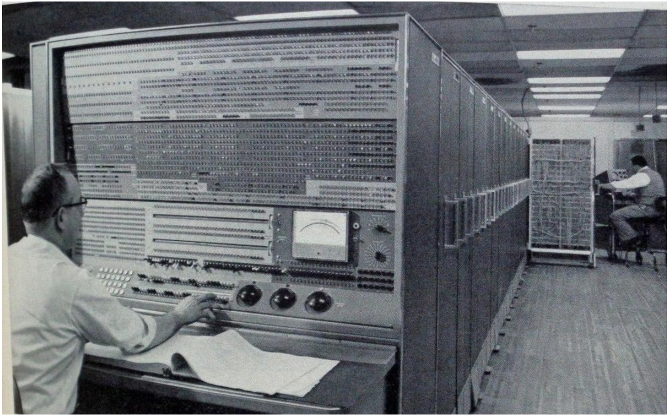

Figure: Los Alamos IBM 7030 STRETCH Computer, 1963

Historical Background

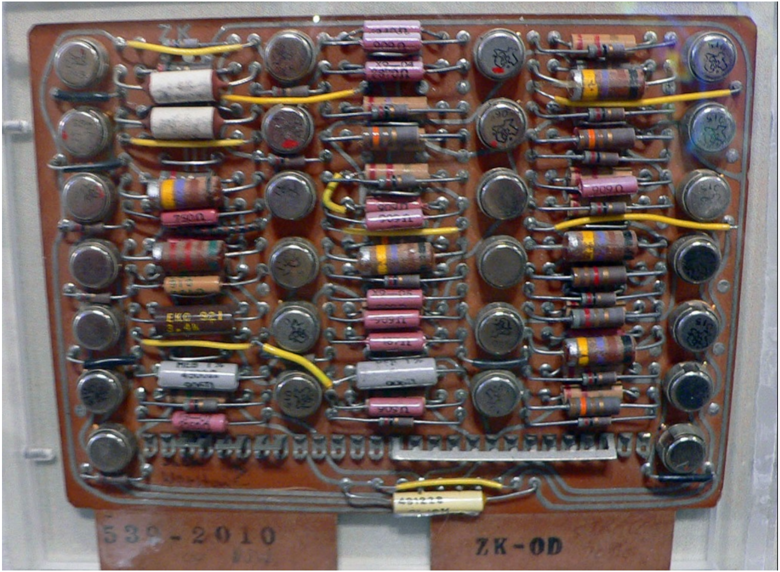

Figure: An IBM 7030 STRETCH circuit board

D.R. Heath-Brown, W.M. Lioen, H.J.J. te Riele (1993):

Implementation on a vector computer, increasing the search space to $|z| \leq |y| \leq |x| \leq 10^8$.

Found solutions for $k \in \{24, 80, 84\}$ and for $k = 2$, a solution not in the Miller-Woollett family:

$$3528875^3 - 3480205^3 - 1214928^3 = -2.$$

Still unsolved at this point: $k \in \{30, 33, 39, 42, 52, 74, 75\}$.

Bremner (1994) rewrites the main equation as

$$(y+t)^3 - y^3 = z^3 + k$$

in the hope that a solution exists with $|t|$ small.

Putting $(X, Y) = ((12tz)^3, 72yt + 36t^2)$, the pair (X, Y) satisfy

$$Y^2 = X^3 - 432t^6 + 12^3kt^3.$$

For $k = 75$, there is an integral point of exactly the correct form for $t = -148$:

$$(X, Y) = (7780938384, 686353676466816).$$

$$453203231^3 - 453203083^3 - 4381159^3 = -75.$$

Other contributors: K. Koyama, D. Bernstein, N. Elkies, S. Huisman **https://math.mit.edu/drew/huisman.txt**

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Booker and Sutherland - 2019

The answer to Mordell's question is YES.

$x = 569936821221962380720$
$y = 569936821113563493509$
$z = 472715493453327032$

then

$$x^3 - y^3 - z^3 = 3.$$

Booker - 2019

$x = 8866128975287528$
$y = 8778405442862239$
$z = 2736111468807040$

then

$$x^3 - y^3 - z^3 = 33.$$

Booker and Sutherland - 2021

$x = 8053873881207597$
$y = 80435758145817515$
$z = 12602123297335631$

then

$$x^3 - y^3 - z^3 = -42.$$

A simple observation

$$x^3 + y^3 + z^3 = k \qquad (|z| < |y| < |x|)$$

with k small (typically) forces $|x| \approx |y|$.

As Bremner did, rewrite (1) as

$$x^3 - y^3 = z^3 + k,$$

and focus on $t = x - y$.

A simple observation continued

Define $t = x - y$, then t is a divisor of

$$x^3 - y^3 = z^3 + k = N_{\mathbb{Q}(k^{1/3})/\mathbb{Q}}(z + k^{1/3}).$$

"Ideally" then,

$$t = N_{\mathbb{Q}(k^{1/3})/\mathbb{Q}}(\alpha) = a^3 + b^3k + c^3k^2 - 3kabc,$$

for some $\alpha = a + bk^{1/3} + ck^{2/3} \in \mathbb{Z}[k^{1/3}]$.

(t can be used to compute z by modular cube roots and lifting, and then x, y by the quadratic formula.)

Obstacles

Integral Basis
For k square-free, $\{1, k^{1/3}, k^{2/3}\}$ is an integral basis.

Class Number
Cheat. I can explain!

Large Search Range
Reduction to searching over b and c only.
(recall $\alpha = a + b \cdot k^{1/3} + c \cdot k^{2/3}$)

Although t may be a product of primes that split in $\mathbb{Z}[k^{1/3}]$, it may not be a norm.

At best, provably, t^h is a norm, where h is the class number.

Experiments show that lt is a norm for some very small factor l.

N.T.Z. Sardari (2018):
(GRH) The least prime in an ideal class $\leq h_K^2 \log(D_K)$.

Consequence: for each $\alpha = a + bk^{1/3} + ck^{2/3}$, $N = N_{K/\mathbb{Q}}(\alpha)$, we test

$$t = N/\gcd(N, l)$$

for small $1 \leq l \leq BOUND$. (currently using $BOUND = 5$)

If

$$t = a^3 + b^3k + c^3k^2 - 3kabc$$

and $|c|$ is large enough wrt t, then

$$|t/c^3| = |(a/c)^3 + (b/c)^3k + k^2 - 3k(a/c)(b/c)|$$

is small, and a/c *may be* close to the* real root, say λ, of

$$P(X) = X^3 - 3k(b/c)X + (k(b/c)^3 + k^2),$$

and $a \approx c\lambda$.

In fact: $a \approx c\lambda \approx -bk^{1/3} - ck^{2/3}$.

THE ALGORITHM

• Create α by looping over certain ranges for b, c
$\alpha = a + bk^{1/3} + ck^{2/3}$

(a is chosen in a small interval containing $-bk^{1/3} - ck^{2/3}$)

• compute the value N of the norm form
$N = a^3 + b^3k + c^3k^2 - 3kabc$

• 'try' $t = N/\gcd(N, l)$ over small integers $l \geq 1$
(i.e. use t, k to compute z by modular cube roots and lifting, and then hope the quadratic formula gives x, y from z, t).

Example 1: $k = 2,\ h = \text{class number} = 1$

$$2 = (6t^3 + 1)^3 - (6t^3 - 1)^3 - (6t^2)^3.$$

Guy asked for solutions not in this family.

$$(x, y, z) = (3528875, 3480205, 1214928)$$

(te Riele and Heath-Brown, $\approx$ 30 hours cpu time in 1990)

$t = 48670$ and

$$t = N_{\mathbb{Q}(2^{1/3})/\mathbb{Q}}(-2 + 23 \cdot 2^{1/3} + 17 \cdot 2^{2/3})$$

($z_1 = 46848 = 2^{1/3} \pmod{t}$, and $z = z_1 + 24 \cdot t$).

Example 2: $k = 7, h = 3$

Small solution: $(x, y, z) = (2, 0, 1)$.

Larger solution:

$$(x, y, z) = (17136136, 13010655, 14144042)$$

produced in the following way:

A small search eventually gives

$$\alpha = -395 + 196 \cdot 7^{1/3} + 18 \cdot 7^{2/3},$$

$N = N(\alpha) = 20627405$.

The method 'tries' $t = N/5 = 4125481$ and succeeds in producing x, y, z.

Example 2': $k = 7, h = 3$ Even larger solution:

$$(x, y, z) = (8940123359, 6427544172, 7657432774)$$

arising from

$$\alpha = 142 + 685 \cdot 7^{1/3} - 60 \cdot 7^{2/3}.$$

EXAMPLE: $k = 74,\ h = 3$

Sander Huisman, 2016: $x^3 - y^3 - z^3 = -74$, where (x, y, z)

$$= (284650292555885, 283450105697727, 66229832190556)$$

$t = x - y = 1200186858158 \neq N_{K/\mathbb{Q}}(\gamma)$ for any $\gamma \in \mathcal{O}$, **but**
$2t = N_{K/\mathbb{Q}}(a + b \cdot 74^{1/3} + c \cdot 74^{2/3})$ for (a, b, c) in

$\{(-56994, -20860, 8205), (33454, 7214, 2039),$
$(8592, -7232, 1295), (213402, 51340, 12137),$
$(49068, -3020, -2041), (4498468, 1071596, 255239),$
$(1830228, 435940, 103799), (10218, -3854, 539),$
$(113820, -39256, 2895)\}$

Example 3: $k = 74, \; h = 3$

(a, b, c)	$floor(-b \cdot 74^{1/3} - c \cdot 74^{2/3})$	
$(-56994, -20860, 8205)$	-57044	$(**)$
$(33454, 7214, 2039)$	-66225	
$(8592, -7232, 1295)$	7536	
$(213402, 51340, 12137)$	429469	
$(49068, -3020, -2041)$	48653	$(*)$
$(4498468, 1071596, 255239)$	8997770	
$(1830228, 435940, 103799)$	-3659786	
$(10218, -3854, 539)$	6679	
$(113820, -39256, 2895)$	113782	$(**)$

Point: Search space for a is almost completely eliminated!

Example 4: $k = 109$ ($h = 3$)

Searching $\alpha = a + b \cdot k^{1/3} + c \cdot k^{2/3}$ can give very erratic and unpredictable results. Here we search with $|a|, |b|, |c| < 100$.

$\alpha = -56 - 16 \cdot 109^{1/3} + 90 \cdot 109^{2/3}$ gives the solution $(X, Y, Z) = (5, -2, -2)$,

$\alpha = -5 + 48 \cdot 109^{1/3} + 29 \cdot 109^{2/3}$ gives the solution $(X, Y, Z) = (-576, 514, 381)$.

HOWEVER, $\alpha = -7 + 64 \cdot 109^{1/3} + 8 \cdot 109^{2/3}$ gives the solution

$$(X, Y, Z) = (-69946755, 44275762, 63448236).$$

Example 5: $k = 33,\ h = 1$

$x = 8866128975287528, y = 8778405442862239$
$z = 2736111468807040, t = x - y = 87723532425289$

$$t = N_{\mathbb{Q}(33^{1/3})/\mathbb{Q}}(102406 + 255182 \cdot 33^{1/3} - 89507 \cdot 33^{2/3})$$

Note: $-255182 \cdot 33^{1/3} + 89507 \cdot 33^{2/3} = 102367.7373...$

Cost: b: $\approx 5 \cdot 10^5$, c: $\approx 2 \cdot 10^5$, a:$\approx 10^2$, lift factor (31):≈ 50
Total Search Space $\approx 5 \cdot 10^{14}$ (Booker took about 10^{16})

Example 6: $k = 42, \ h = 3$

$x = 80538738812075974, y = 80435758145817515$
$z = 12602123297335631, t = x - y = 102980666258459$

t is **not** a norm, however

$$2t = N_{\mathbb{Q}(42^{1/3})/\mathbb{Q}}(-473180 - 76667 \cdot 42^{1/3} + 61231 \cdot 42^{2/3})$$

Note: $76667 \cdot 42^{1/3} - 61231 \cdot 42^{2/3} = -473343.0184...$

Cost: b: $\approx 2 \cdot 10^5$, c: $\approx 2 \cdot 10^5$, a:$\approx 5 \cdot 10^2$, lift factor (122):$\approx 10^2$
Total Search Space $\approx 2 \cdot 10^{15}$ (B and S took about 10^{18})

$x = 569936821221962380720$
$y = 569936821113563493509$
$z = 472715493453327032$
then

$$x^3 - y^3 - z^3 = 3. \qquad \text{(Booker and Sutherland, 2019)}$$

In this case, $t = x - y = 108398887211$.

$t = N(\alpha)$ with $\alpha = a + b \cdot 3^{1/3} + c \cdot 3^{2/3}$ and (a, b, c) one of

$$(5603, 1612, 1156), (-2386, -143, 2432), (3824, -2135, 992).$$

Two problems arise: approximation and lifting.
$(s = (z - z_1)/t = 4360888)$

The factorization (recall $t = N_{K/\mathbb{Q}}(a + b\alpha + c\alpha^2)$)

$$(a + b\alpha + c\alpha^2)(d + e\alpha + f\alpha^2) = z + \alpha$$

gives

$$d = \tfrac{1}{t}(z(a^2 - kbc) + (kb^2 - kac))$$
$$e = \tfrac{1}{t}(z(kc^2 - ac) + (a^2 - kbc))$$
$$f = \tfrac{1}{t}(z(b^2 - ac) + (kc^2 - ab)).$$

Put $z = z_1 + ts$ with $z_1 \equiv -(kb^2 - kac)(a^2 - 3bc)^{-1} \pmod{t}$,

to get 'cubic in s = quadratic in Y':

$$N_{K/\mathbb{Q}}(d + e\alpha + f\alpha^2) = d^3 + ke^3 + k^2 f^3 - 3kdef = 3Y^2 + 3Yt + t^2.$$

Point: Let $r = 6Y + 3t$, then (s, r) is an integral point on

$$r^2 = A_0 s^3 + A_1 s^2 + A_2 s + A_3,$$

with $A_i = A_i(a, b, c) \quad (i = 1, 2, 3)$.

$k = 3 : t = 108398887211 = N_{K/\mathbb{Q}}(5603 + 1612\alpha + 1156\alpha^2)$,

$d = 25803193s + 20709780,\ e = -5023028s - 4031509,$
$f = -3878524s - 3112924.$

and $s = 4360888$ is the x-coordinate of an integral point on

$r^2 = 14100382498299719230225 2s^3$
$+339511260630207004599744s^2$
$+2724935443148718714562 56s$
$+7650633287640319405761.$

Booker and Sutherland: "Several hundred core-years".

The method above
Time to find an (a, b, c) curve with **one** computable integral point) $\times$ (cost to find that point on the curve)

Search in the cube $0 \leq a, b, c < 10^4$, and test the x-coordinates up to 10^7. (c is the outer loop)

time $\approx (10^4)^3 \times .0025$ seconds with **hyperellratpoints** (M. Stoll)

Expected time: 80 core years.

Actual time to find the solution: $\approx$ 9.16 core-years!
(3.5 days on a 1000-core dedicated machine)