Online Number Theory Seminar

24 February 2023. – 17:00-17:50

Florian Luca: Y-coordinates of Pell equations in binary recurrences

Let $d > 1$ be an integer which not a square and (X_n, Y_n) be the nth solution of the Pell equation $X^2 - dY^2 = \pm 1$. Given an interesting set of positive integers U, we ask how many positive integer solutions n can the equation $Y_n \in U$ have. Under mild assumptions on U (for example, when $1 \in U$ and U contains infinitely many even integers), the equation $Y_n \in U$ has two solutions n for infinitely many d. We show that this is best possible whenever U is the set of values of a binary recurrent sequence $\{u_m\}_{m\geq 1}$ with real roots and d is large enough (with respect to U). We also treat the cases when U is one of the sets $\{2^n - 1 : n \geq 1\}$, $\{F_n : n \geq 1\}$ and $\{L_n : n \geq 1\}$, where F_n and L_n are the nth Fibonacci and Lucas numbers. For example, $Y_n = 2^m - 1$ has at most two positive integer solutions (n, m) for all d and each of the equations $Y_n = F_m$ or $Y_n = L_m$ has exactly two solutions (n, m) except for $d = 2$, in which case it has exactly three solutions both when Fibonacci or Lucas numbers are involved. The proofs use linear forms in logarithms.