An introduction to Hopf-Galois theory

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Now, $K[G]$ is a K-Hopf algebra (a K-vector space with some additional structure).

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- A Hopf-Galois extension might have several Hopf-Galois structures. It is possible that the same Hopf algebra endowed with different actions give rise to different Hopf-Galois structures.
- 'Fundamental' theorem of Hopf-Galois theory: If L/K is H-Galois, there is an injective map from the Hopf subalgebras of H to the intermediate fields of L/K.

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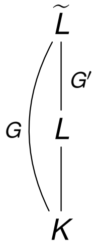

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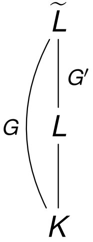

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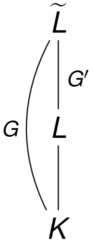

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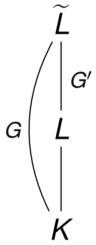

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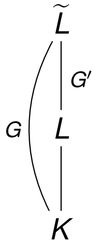

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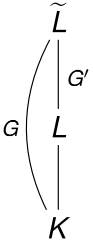

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Theorem (Greither-Pareigis theorem, 1986)

There is a bijective correspondence between the following:

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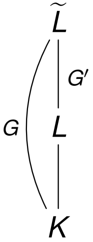

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A subgroup $N \leq \mathrm{Perm}(X)$ is regular if

$$\forall x, y \in X \, \exists! \eta \in N : \; \eta(x) = y.$$

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  - A separable extension with Burnside degree admits at most one Hopf-Galois structure.

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Definition

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1. $(B, \cdot)$ *and* $(B, \circ)$ *are groups.*
2. *For each* $a, b, c \in B$, $a \circ (b \cdot c) = (a \circ b) \cdot a^{-1} \cdot (a \circ c)$.

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This link led to new questions and answers in both areas and currently constitutes a very active research field.

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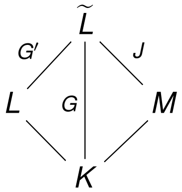

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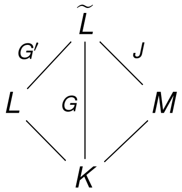

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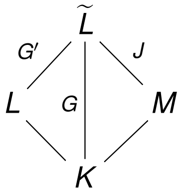

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Usual strategy: Translate properties from $\widetilde{L}/M$ to L/K.

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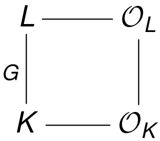

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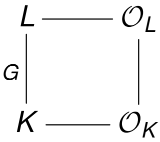

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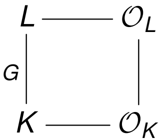

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The issue is that $\mathcal{O}_L$ is not $\mathcal{O}_K[G]$-free for a bunch of extensions L/K.

Definition (Leopoldt, 1959)

Let L/K be a Galois extension of number or p-adic fields with group G. The associated order of $\mathcal{O}_L$ in $K[G]$ is defined as

$$\mathfrak{A}_{L/K} = \{\lambda \in K[G] \mid \lambda \cdot x \in \mathcal{O}_L \text{ for all } x \in \mathcal{O}_L\}.$$

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$$\mathfrak{A}_{L/K} = \{\lambda \in K[G] \mid \lambda \cdot x \in \mathcal{O}_L \text{ for all } x \in \mathcal{O}_L\}.$$

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Byott (1997): There is a Galois extension L/K of p-adic fields such that $\mathcal{O}_L$ is not $\mathfrak{A}_{L/K}$-free but $\mathcal{O}_L$ is $\mathfrak{A}_H$-free in some other Hopf-Galois structure H on L/K.

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Proposition

$L = \mathbb{Q}(\sqrt{a(d+b\sqrt{d})})$, where:

- *$a \in \mathbb{Z}$ is odd square-free and $b \in \mathbb{Z}_{>0}$.*
- *$d = b^2 + c^2$ for some $c \in \mathbb{Z}_{>0}$ and d is square-free.*
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Theorem

Let $L/\mathbb{Q}$ *be a cyclic quartic extension defined by* $a, b, c, d \in \mathbb{Z}$ *as before. Let* H *be its non-classical Hopf-Galois structure. The following are equivalent:*

- $\mathcal{O}_L$ *is* $\mathfrak{A}_H$*-free.*
- *The quadratic form* $[b, 2c, -b]$ *represents* 1.
- *The Pell equation* $x^2 - dy^2 = b$ *has some solution* (x, y) *such that* b *divides* $x - cy$.

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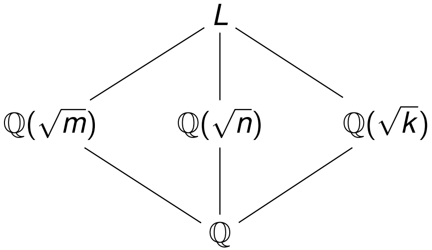

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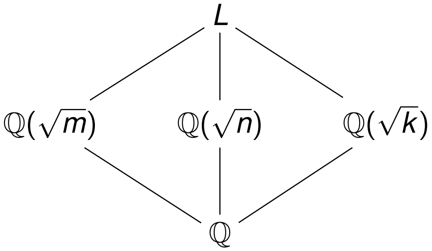

There are three non-classical Hopf-Galois structures H_m, H_n, H_k on L/K, each one depending on an intermediate field.

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Mod 4		$\mathcal{O}_L$ as $\mathfrak{A}_{H_i}$-module		
m	n	H_m	H_n	H_k
1	1	Free $\iff$ $\exists x, y \in \mathbb{Z}$: $x^2 + my^2 = \pm 2d$	Free $\iff$ $\exists x, y \in \mathbb{Z}$: $x^2 + ny^2 = \pm 2d$	Free $\iff$ $\exists x, y \in \mathbb{Z}$: $x^2 + ky^2 = \pm 2\frac{n}{d}$
1	$\neq 1$	Free $\iff$ $\exists x, y \in \mathbb{Z}$: $x^2 + my^2 = \pm 2d$	Not free	Not free
3	2	Free $\iff$ $\exists x, y \in \mathbb{Z}$: $x^2 + my^2 = \pm 4d$	Free $\iff$ $\exists x, y \in \mathbb{Z}$: $x^2 + ny^2 = \pm 2d$	Free $\iff$ $\exists x, y \in \mathbb{Z}$: $x^2 + ky^2 = \pm 2\frac{n}{d}$

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Theorem

Let $L = \mathbb{Q}(\sqrt[n]{a})$ with $a \in \mathbb{Q}$. Assume that $L \cap \mathbb{Q}(\zeta_n) = \mathbb{Q}$ and that $\mathcal{O}_L = \mathbb{Z}[\sqrt[n]{a}]$. Then there is some Hopf-Galois structure H on $L/\mathbb{Q}$ such that $\mathcal{O}_L$ is $\mathfrak{A}_H$-free.

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Thank you for your attention