An open source implementation for solving S-unit equations

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Outline

1. Introductions / Background / History
2. A Selection of Applications
3. How the Solver Works
4. What's Good / What's Bad / What's Next

Outline

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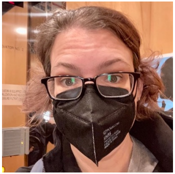

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S-unit equation Collaborators

With assistance from:

Norman Danner, Bjorn Poonen, David Roe, Andrew Sutherland, . . .

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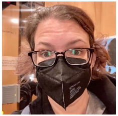


And building on the past work of:

Baker, Baker-Wüstholz, Bremner, Brumer, de Weger, Evertse-Győry, Gelfond, Győry, Győry-Yu, Koutsianas, Lenstra-Lenstra Lovász, Mahler, Malmskog-R., Merriman-Smart, Pethö-de Weger, SageMath Developers, Schneider, Siegel, Smart, Tzanakis-de Weger, Wildanger, Yu, . . .

Independent Efforts

The community is fortunate to have *multiple* efforts to solve unit equations under active development.

- Joint work of von Känel and Matschke on arithmetic of elliptic curves with good reduction outside S (includes S-unit equations over $\mathbb{Q}$, S-integral points on curves, Thue-equations, ...)
  More Information

- Benjamin Matschke has a general S-unit solver, currently in development.
  More Information

General Unit Equation In Two Variables

K	a number field of degree d	
Γ_0, Γ_1	finitely generated subgroups of $K^\times$	$\boldsymbol{\Gamma} := \Gamma_0 \times \Gamma_1$
τ_0, τ_1	variables (view $\tau_i \in \Gamma_i$)	$\boldsymbol{\tau} := (\tau_0, \tau_1) \in \boldsymbol{\Gamma}$
α_0, α_1	fixed elements of $K^\times \times K^\times$	$\boldsymbol{\alpha} := (\alpha_0, \alpha_1)$

$$\boldsymbol{\alpha} \cdot \boldsymbol{\tau} = \alpha_0\tau_0 + \alpha_1\tau_1$$

Problem

Determine the set $T = \{\boldsymbol{\tau} \in \boldsymbol{\Gamma} : \boldsymbol{\alpha} \cdot \boldsymbol{\tau} = 1\}$.

An Incomplete History

$$T = \{\tau \in \mathbf{\Gamma} : \alpha \cdot \tau = 1\}$$

1921 (Siegel) $\#T < \infty$ for any number field K, $\Gamma_0 = \Gamma_1 = \mathcal{O}_K^\times$.

1933 (Mahler) $\#T < \infty$ for $K = \mathbb{Q}$, $\Gamma_0 = \Gamma_1 = \mathbb{Z}[\frac{1}{p_1}, \ldots, \frac{1}{p_r}]^\times$.

1934 (Gelfond, Schneider) For $\alpha, \beta \in \overline{\mathbb{Q}}$ with $\alpha \neq 0, 1$ and $\beta \notin \mathbb{Q}$, $\alpha^\beta \in \mathbb{C} - \overline{\mathbb{Q}}$.

1950 (Parry) $\#T < \infty$ for any number field K, $\Gamma_i = \mathcal{O}_{K,S}^\times$, any finite S.

1960 (Lang) $\#T < \infty$ for any K with char $K = 0$, any f.g. $\Gamma_i \leq K^\times$

1967 (Baker) For $\beta_i \in \overline{\mathbb{Q}}$ with $\{\log \beta_i\}$ $\mathbb{Q}$-independent, and for any nonzero linear form $L \in \overline{\mathbb{Q}}[\mathbf{X}]$,

$$|L(\log \beta_1, \ldots, \log \beta_r)| > H(L)^{-C}, \qquad C: \text{ effective}$$

1968 (Bremner) For $\alpha_i \in \overline{\mathbb{Q}}_p^\wedge$, $\mathbb{Q}$-independence of $\{\log_p \alpha_i\}$ implies $\overline{\mathbb{Q}}$-independence.

An Incomplete History

$$T = \{\tau \in \mathbf{\Gamma} : \alpha \cdot \tau = 1\}$$

1974 (Győry) First explicit bounds on solutions in T.

1984 (Evertse) Bound on $\#T$ when $\Gamma_0 = \Gamma_1 = \mathcal{O}_{K,S}^{\times}$.

1985 (Evertse-Győry) Explicit bounds on $\#$ of solutions in $\mathcal{O}_{K,S}^{\times}$ to Thue eqns. $F(\mathbf{X}) = \beta$.

1988 (Evertse-Győry-Stewart-Tijdeman) Fix $K, \Gamma \leq K^{\times}$. For $\alpha \in \Gamma^2$, define $N(\alpha) := \#\{\tau \in \Gamma^2 : \alpha \cdot \tau = 1\}$.

There exist only finitely many α with $N(\alpha) > 2$.

An Incomplete History

$$T = \{\tau \in \mathbf{\Gamma} : \alpha \cdot \tau = 1\}$$

1988 (Yu) Fix $\mathfrak{p} \subseteq \mathcal{O}_K$. For $\rho_i \in K^\times$ with $\operatorname{ord}_\mathfrak{p} \rho_i = 0$, either $\rho_0^{b_0}\rho_1^{b_1}\cdots\rho_t^{b_t} = 1$, or

$$\operatorname{ord}_\mathfrak{p}\left(\rho_0^{b_0}\rho_1^{b_1}\cdots\rho_t^{b_t} - 1\right) < C, \qquad C : \text{effective}$$

1993 (Baker-Wüstholz) Improvements to bounds in (Baker, 1967).

1996 (Beukers-Schlickewei) Bounds for $\#T$ in terms of $\operatorname{rank}_\mathbb{Z} \Gamma_i$ only.

2006 (Győry-Yu) For $\Gamma_0 = \Gamma_1 = \mathcal{O}_{K,S}^\times$ and $s = \#S$, any $\tau \in T$ satisfies

$$h(\tau_i) < (16ds)^{2s+6}\left(1 + \frac{\max\{1, \log R_S\}}{\max\{1, \log P_S\}}\right) \cdot \max_i\{h(\alpha_i)\}$$

An Incomplete History

$$T = \{\tau \in \boldsymbol{\Gamma} : \alpha \cdot \tau = 1\}$$

2016 (von Känel-Matschke) For $K = \mathbb{Q}$, $\Gamma_i = \mathcal{O}_{K,S}^{\times}$, can obtain bounds without methods of Baker, Yu. Solutions induce elliptic curves of specific conductor.

2019 (Győry) Best known bounds for $\Gamma_0 = \Gamma_1 = \mathcal{O}_{K,S}^{\times}$. Formulas avoid self-exponential factors, e.g., s^s.

Outline

Application: Asymptotic Fermat

$$\mathcal{C}_p \colon x^p + y^p + z^p = 0, \qquad abc \neq 0, p > 3 \text{ prime}$$

Theorem (Wiles)

$\#\mathcal{C}_p(\mathbb{Q}) = \varnothing$.

- $\#\mathcal{C}_p(K) < \infty$ by Faltings.
- We say K satisfies **asymptotic Fermat** if $\mathcal{C}_p(K) = \varnothing$ for $p > B_K$.

Theorem (Freitas-Siksek)

There exists a family of real quadratic number fields of density at least $\frac{5}{6}$ which satisfy asymptotic Fermat.

Application: Asymptotic Fermat

$$S = \{\mathfrak{p} : \mathfrak{p} \mid 2 \text{ and } f_{\mathfrak{p}} = 1\}.$$

Theorem (Freitas-Siksek)

Suppose K is totally real, and suppose $[K : \mathbb{Q}]$ is odd or $S \neq \varnothing$. If for every solution $\tau \in T$, $\operatorname{ord}_{\mathfrak{p}} \tau_i \leq 4 \operatorname{ord}_{\mathfrak{p}} 2$, then K satisfies asymptotic Fermat.

Theorem (AKMRVW)

Suppose $[K : \mathbb{Q}] = 3$, K is totally real, 2 is totally ramified in K, and $|\Delta_K| \leq 2000$. Then K satisfies asymptotic Fermat.

Application: Cubic Ramanujan-Nagell

Theorem (Nagell, 1948)

If $x, n \in \mathbb{Z}^{\geq 0}$ *satisfy* $x^2 + 7 = 2^n$*, then* $x \in \{1, 3, 5, 11, 181\}$.

Cubic Ramanujan-Nagell equations: $x^3 + p^k = q^n$.

For $p = 3$ and fixed q, solutions (x, k, n) may be found by solving the S-unit equation over $\mathbb{Q}(\sqrt[3]{3})$ with $S = \{\mathfrak{p} : \mathfrak{p} \mid 3q\} \cup M_K^\infty$.

Theorem (AKMRVW)

For $q < 500$*, there are exactly* 11 *solutions* (x, k, n, q) *to* $x^3 + p^k = q^n$*, and all have* $n = 1$.

Application: Curves with bad reduction at one prime

- Suppose $C \to \mathbb{P}^1$ is a cyclic degree p cover and C has good reduction outside p.

- Differences of branch points, $\alpha_i - \alpha_j$, must be S-units.

$$(\alpha_i - \alpha_j) + (\alpha_j - \alpha_k) = \alpha_i - \alpha_k$$
$$\frac{\alpha_i - \alpha_j}{\alpha_i - \alpha_k} + \frac{\alpha_j - \alpha_k}{\alpha_i - \alpha_k} = 1.$$

- $K = \mathbb{Q}(\{\alpha_i\})$ has $\Delta_K = \pm p^m$.

Theorem (Smart, 1994)

Every genus 2 curve $C/\mathbb{Q}$ with good reduction away from 2 is isomorphic over $\mathbb{Q}$ to a curve appearing in an explicit finite list.

Theorem (Malmskog, R., 2014)

Up to $\mathbb{Q}$-isomorphism, there are exactly 63 Picard curves $C/\mathbb{Q}$ with good reduction away from 3 and a complete list of representative curves has been produced.

Many Other Applications

- Enumerative problems, e.g. C/K with good reduction outside S
- Effective finiteness for binary forms (Evertse-Győry)
- Effective results for discriminant form, index form equations. (Győry)
- Effective methods on deciding monogeneity ($\exists ?\, \alpha$ s.t. $\mathcal{O}_K = \mathbb{Z}[\alpha]$) in number fields, and for determining all integral bases (Győry)
- Strong and effective bounds towards abc-Conjecture (Győry)
- among others . . .

Outline

Notation

For the remainder: $\alpha = (1,1)$, $\Gamma_0 = \Gamma_1 = \mathcal{O}_{K,S}^\times$.

- K, a number field, $\quad d_K := [K : \mathbb{Q}]$, $\qquad w := \#\mu_K$.
- $S = S_{\text{fin}} \cup M_K^\infty$, a finite set of places (incl. all infinite places)

$$S_{\text{fin}} = \{\mathfrak{p}_1, \mathfrak{p}_2, \ldots, \mathfrak{p}_s\},$$
$$M_K^\infty = \{\mathfrak{p}_{s+1}, \ldots, \mathfrak{p}_{s+r+1}\}.$$

- $\mathcal{O}_{K,S}^\times$, the group of S-units in $K^\times$

$$\mathcal{O}_{K,S}^\times = \langle \rho_0 \rangle \times \langle \rho_1, \ldots, \rho_t \rangle \cong \frac{\mathbb{Z}}{w\mathbb{Z}} \times \mathbb{Z}^t$$

Shorthand : $\qquad \boldsymbol{\rho} = (\rho_0, \rho_1, \ldots, \rho_t)$.

Exponent Vectors

- $A_{K,S} := \frac{\mathbb{Z}}{w\mathbb{Z}} \times \mathbb{Z}^t$, $\qquad \Phi_\rho : A_{K,S} \xrightarrow{\cong} \mathcal{O}_{K,S}^\times$,

$$\mathbf{a} = (a_0, a_1, \ldots, a_t) \mapsto \rho^{\mathbf{a}} := \rho_0^{a_0} \rho_1^{a_1} \cdots \rho_t^{a_t}.$$

- Elements $\mathbf{a} \in A_{K,S}$ are called **exponent vectors**.

$$|\mathbf{a}| := \max\{|a_i| : 0 \leq i \leq t\}.$$

- $X_{K,S} := \{x \in \mathcal{O}_{K,S}^\times : 1 - x \in \mathcal{O}_{K,S}^\times\}$, $\qquad E_{K,S} := \Phi_\rho^{-1} X_{K,S}$.

- Solving $\tau_0 + \tau_1 = 1$ is equivalent to finding $E_{K,S}$ inside $A_{K,S}$.

Outline of Algorithm

1. Use bounds on linear forms in logarithms (Baker-Wüstholz, Yu), determine K_0 such that $\mathbf{a} \in E_{K,S} \implies |\mathbf{a}| \leq K_0$.
    - quick (run time < 1 second)
    - K_0 hopelessly large
2. Run a LLL argument to deduce a better bound $|\mathbf{a}| \leq K_1$.
    - quick (run time in seconds)
    - effective ($K_1 \approx (\log K_0)^c$)
    - *not* guaranteed to work
    - *requires* a known K_0
3. Extract $E_{K,S}$ from search space of size $\approx w(2K_1)^t$ by sieve
    - slow and expensive (time and memory)
    - sensitive to which primes $q \in \mathbb{Z}$ split completely in K

Finding the initial bounds

Suppose (τ_0, τ_1) is a solution with $\tau_i = \rho^{\mathbf{b}_i}$, $\quad B := |\mathbf{b}_0| \geq |\mathbf{b}_1|$

- Loop over $\ell \in \{1, 2, \dots, t+1\}$:
  - Suppose $|\tau_0|_{\mathfrak{p}_\ell} = \min \left\{ |\tau_0|_{\mathfrak{p}_k} : 1 \leq k \leq t+1 \right\}$.
  - By standard argument[1] we have $|\tau_0|_{\mathfrak{p}_\ell} \leq \exp\left(-c_3 B\right)$.
  - Calculate explicit bound $K_0(\ell)$ satisfying $B \leq K_0(\ell)$.
    - Case I: $\mathfrak{p}_\ell \in M_K^\infty$
    - Case II: $\mathfrak{p}_\ell \in S_{\text{fin}}$
- Set $K_0 := \max\{K_0(\ell) : 1 \leq \ell \leq t+1\}$.

[1] Smart, The solution to TCDF equations, *Math. Comp.*, 1995.

Initial Bounds – Case I: $\mathfrak{p}_\ell \in M_K^\infty$

- $\mathfrak{p}_\ell$ corresp. to some $\sigma_\ell \colon K \to \mathbb{C}$. Work in $\sigma_\ell(K)$.

- τ_0 near 0 $\implies$ τ_1 near 1 $\implies$ $\log \tau_1$ near 0:

$$|\log \tau_1| \leq 2 \exp(-c_{13} B).$$

- But $\log \tau_1 = b_{1,0} \log \rho_0 + b_{1,1} \log \rho_1 + \cdots + b_{1,t} \log \rho_t$!

- (Baker-Wüstholz) $\qquad |\log \tau_1| \geq \exp(-c_{14} \log B)$.

- $\therefore B < a + b \log B \quad \implies \quad B \leq K_0(\ell)$.

Initial Bounds – Case II: $\mathfrak{p}_\ell \in S_{\text{fin}}$

- $|\tau_0|_{\mathfrak{p}_\ell} \leq \exp(-c_3 B) \implies \text{ord}_{\mathfrak{p}_\ell} \tau_0 \geq c_5' B > 0$
- $\text{ord}_{\mathfrak{p}_\ell} \tau_0 > 0 \quad \implies \quad \text{ord}_{\mathfrak{p}_\ell} \tau_1 = 0.$
- Replace ρ with $\mu := (\mu_0, \mu_1, \ldots, \mu_{t-1})$, $\text{ord}_{\mathfrak{p}_\ell} \mu_i = 0$.

$$\tau_1 = \mu^{\mathbf{d}}, \qquad |\mathbf{d}| \leq B.$$

- (Yu) $\text{ord}_{\mathfrak{p}_\ell} \tau_0 < c_8' \log B$
- $\therefore B < b \log B \quad \implies \quad B \leq K_0(\ell)$.

Reduction via LLL - Preliminaries

- Suppose $\mathcal{L} = \mathbb{Z}\langle \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_N \rangle \subseteq \mathbb{R}^N$, $\quad \mathcal{L}^* := \mathcal{L} - \{\mathbf{0}\}$

- For $\mathbf{y} \in \mathbb{R}^N$, $\ell(\mathcal{L}, \mathbf{y}) := \begin{cases} \min\{||\mathbf{x}|| : \mathbf{x} \in \mathcal{L}^*\} & \text{if } \mathbf{y} \in \mathcal{L}, \\ \min\{||\mathbf{x} - \mathbf{y}|| : \mathbf{x} \in \mathcal{L}\} & \text{if } \mathbf{y} \notin \mathcal{L}. \end{cases}$

LLL Theorem

The reduced basis $\mathbf{x}_1, \ldots, \mathbf{x}_N$ produced when the LLL algorithm is applied to $\mathcal{L}$ satisfies $\ell(\mathcal{L}, \mathbf{0}) \geq m_{\mathcal{L},\mathbf{0}} := (\text{constant}) \cdot ||\mathbf{x}_1||$.

- Idea: Build integer lattice $\mathcal{L}$ from $\tau_1 = \rho^{\mathbf{b}_1}$. If $\ell(\mathcal{L}, \mathbf{y})$ is large, the bound K_0 may be replaced with $K_1(\ell) \ll K_0$.

- Again depends on the "extremal" place $\mathfrak{p}_\ell$: $K_1 := \max\{K_1(\ell)\}$.

Case: $\mathfrak{p}_\ell$ complex

- Write $\log \rho_j = \kappa_j + \lambda_j\sqrt{-1}, \qquad \kappa_j, \lambda_j \in \mathbb{R}$.
- Pick C large. (On the order of K_0.)
- Take $\mathcal{L}$ spanned by columns $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{t+1}$ of

$$\begin{pmatrix} 1 & & 0 & 0 & 0 \\ & \ddots & & \vdots & \vdots \\ 0 & & 1 & 0 & 0 \\ [C\kappa_1] & \cdots & [C\kappa_{t-1}] & [C\kappa_t] & 0 \\ [C\lambda_1] & \cdots & [C\lambda_{t-1}] & [C\lambda_t] & [C\cdot\frac{2\pi}{w}] \end{pmatrix}$$

- Apply LLL and compute the bound $m_{\mathcal{L},\mathbf{0}}$.
- Take $\mathbf{y} = b_{1,1}\mathbf{v}_1 + \cdots + b_{1,t}\mathbf{v}_t + b_{1,0}\mathbf{v}_{t+1} \in \mathcal{L} \qquad (\tau_1 = \rho^{\mathbf{b}_1})$

Case: $\mathfrak{p}_\ell$ complex

$$\mathbf{y} = b_{1,1}\mathbf{v}_1 + \cdots + b_{1,t-1}\mathbf{v}_{t-1} + b_{1,t}\mathbf{v}_t + b_{1,0}\mathbf{v}_{t+1}$$

$$= \begin{pmatrix} b_{1,1} & b_{1,2} & \cdots & b_{1,t-1} & \Phi_1 & \Phi_2 \end{pmatrix}^{\mathsf{T}}$$

- By design, $\Phi_1 + \Phi_2\sqrt{-1}$ is "close" to $C \log \tau_1$.

$$\begin{aligned} m_{\mathcal{L},\mathbf{0}}^2 \leq ||\mathbf{y}||^2 &= \sum_{j=1}^{t-1} b_{1,j}^2 + \left|\Phi_1 + \Phi_2\sqrt{-1}\right|^2 \\ &\leq tK_0^2 + (twK_0 + C\log\tau_1)^2 \\ &\leq tK_0^2 + (twK_0 + 2C\exp(-c_{13}B))^2 \end{aligned}$$

- If $m_{\mathcal{L},\mathbf{0}} \gg twK_1^2$, this is a stronger constraint on B:

$$\therefore B \leq K_1(\ell) \approx \frac{1}{c_{13}} \log\left(\frac{2C}{\sqrt{m_{\mathcal{L},\mathbf{0}} - tK_0} - twK_0}\right).$$

Case: $\mathfrak{p}_\ell$ complex

If $m_{\mathcal{L},\mathbf{0}} \gg twK_1^2$,

$$B \leq K_1(\ell) \approx \frac{1}{c_{13}} \log\left(\frac{2C}{\sqrt{m_{\mathcal{L},\mathbf{0}} - tK_0} - twK_0}\right).$$

- WHILE $m_{\mathcal{L},\mathbf{0}} < twK_1^2$:
  - $C \leftarrow 2C$ (changes $\mathcal{L}$)
  - Re-run LLL and re-compute $m_{\mathcal{L},\mathbf{0}}$
- Record new exponent bound $K_1(\ell)$.
- If so inclined, replaced K_0 with $K_1(\ell)$ and run again!

The approach when $\mathfrak{p}_\ell$ is real is similar – slightly different lattice $\mathcal{L}$.

Case: $\mathfrak{p}_\ell$ finite

- Same idea, but we use a different lattice $\mathcal{L}$.

$$\Delta := \log_p \tau_1 = \log_p \mu_0 + \sum_i d_i \log_p \mu_i$$

- $K_{\mathfrak{p}_\ell} = \mathbb{Q}_p(\theta)$. Express Δ in the power basis:

$$\Delta = \Delta_0 + \Delta_1\theta + \Delta_2\theta^2 + \cdots + \Delta_{n-1}\theta^{n-1}$$

- By power series expansion of $\log_p$:

$$\Delta_k = a_{0,k} + \sum_{j=1}^{t-1} d_j a_{j,k}, \qquad a_{j,k} \in \mathbb{Q}_p$$

- By appropriate scaling,

$$\lambda^{-1}\Delta_k = \kappa_{0,k} + \sum_{j=1}^{t-1} d_j \kappa_{j,k}, \qquad \kappa_{j,k} \in \mathbb{Z}_p$$

Case: $\mathfrak{p}_\ell$ finite

- Pick a large u. For $\mathcal{L}$, we take the columns of $\begin{pmatrix} I_{t-1} & O \\ \kappa & p^u I_n \end{pmatrix}$,

  where $\kappa := \begin{pmatrix} \kappa_{1,0} & \cdots & \kappa_{t-1,0} \\ \vdots & & \vdots \\ \kappa_{1,n-1} & \cdots & \kappa_{t-1,n-1} \end{pmatrix}$ "(mod p^u)".

- $\mathbf{y} = -(0, \cdots, 0, \kappa_{0,0}, \cdots, \kappa_{0,n-1})^\intercal$ ("mod p^u") Note: $\mathbf{y} \notin \mathcal{L}$

- Similar to complex case: if $\ell(\mathcal{L}, \mathbf{y}) > t^{\frac{1}{2}} K_0$, we may conclude

$$B < K_1(\ell) \approx (\text{constant}) \cdot u$$

- (If not, $u \leftarrow 2u$ and rerun LLL, etc.)

After the LLL Reduction

- This LLL step is *AMAZING* . . .

$$K_0 \approx 10^{300} \quad \xrightarrow{\text{LLL}} \quad K_1 \approx 4000 \quad \xrightarrow{\text{LLL}} \quad K_1 \approx 300$$

- . . . but not amazing *ENOUGH*, e.g.:

$$K_1 \approx 300, t = 6, w = 2 \quad \Longrightarrow \quad w(2K_1 + 1)^t \approx 4.7 \times 10^{16}.$$

- Need a sieving procedure to execute the final search efficiently.

Sieving against primes away from S

- $d = [K : \mathbb{Q}]$, q splits completely in K, $q \notin \mathfrak{p}$ for all $\mathfrak{p} \in S_{\text{fin}}$.

- $q\mathcal{O}_K = \mathfrak{q}_0\mathfrak{q}_1 \cdots \mathfrak{q}_{d-1}$, each $\mathbb{F}_{\mathfrak{q}_i} \cong \mathbb{F}_q$.

- Set $A_{K,S,q-1} := (\mathbb{Z}/w\mathbb{Z}) \times (\mathbb{Z}/(q-1)\mathbb{Z})^t$.

$$\pi_{q-1} \colon A_{K,S} \to A_{K,S,q-1}$$

- **residue field vector**: For $\tau \in \mathcal{O}_{K,S}^\times$,

$$\mathrm{rfv}_q\, \tau := (\tau + \mathfrak{q}_0, \tau + \mathfrak{q}_1, \ldots, \tau + \mathfrak{q}_{d-1}) \in \mathbb{F}_q^d.$$

Sieving against primes away from S

$$\operatorname{rfv}_q \tau := (\tau + \mathfrak{q}_0, \tau + \mathfrak{q}_1, \ldots, \tau + \mathfrak{q}_{d-1}) \in \mathbb{F}_q^d.$$

- Suppose $\mathbf{a}, \mathbf{b} \in A_{K,S}$ with $\rho^{\mathbf{a}} + \rho^{\mathbf{b}} = 1$. Then:
    - $\operatorname{rfv}_q \rho^{\mathbf{a}} + \operatorname{rfv}_q \rho^{\mathbf{b}} = \mathbf{1} := (1, 1, \ldots, 1) \in \mathbb{F}_q^d$.
    - No entry of $\operatorname{rfv}_q \rho^{\mathbf{a}}$ is 0 or 1.
    - $\operatorname{rfv}_q \rho^{\mathbf{a}}$ is determined by $\pi_{q-1}(\mathbf{a})$.

- rfv_q is not surjective: there might not exist $\mathbf{b}$ such that

$$\operatorname{rfv}_q \rho^{\mathbf{a}} = \mathbf{1} - \operatorname{rfv}_q \rho^{\mathbf{b}}.$$

- Using several q gives a large collection of congruences that the exponent vectors in $\rho^{\mathbf{a}} + \rho^{\mathbf{b}} = 1$ must satisfy.

Outline

What's Good / What's Bad / What's Next

- The Good
  - 100% public and open source
  - Integrated into CoCalc/Sage

In [3]:
```
from sage.rings.number_field.S_unit_solver import *
K.<a> = NumberField(x^2 + 7)
S = K.primes_above(14)
OKSx = K.S_unit_group(S=S)
solve_S_unit_equation(K, S, prec=200)
```

  - Reviewed many times (us, referee(s), Sage submission)
  - Well documented (arXiv:1903.00977, now in *Simons Symposia*)

- The Bad
  - The case $K = \mathbb{Q}$ is handled poorly
  - The current sieve is slow
  - Currently restricted to $\alpha = (1, 1)$, $\mathbf{\Gamma} = \mathcal{O}_{K,S}^\times \times \mathcal{O}_{K,S}^\times$

- Next Steps
  - Many improvements within reach

Planned Improvements

- Best known height bounds on solutions (Győry, 2019) may improve Step 1 bounds
- Taking $K_1 = \max K_1(\ell)$ is inefficient. We can track bounds on each exponent and shrink the search space.
- Use ideas of Smart and Wildanger to eliminate "extreme corners" of the search space.
- Replace the existing sieve with a faster exhaustive search based on Fincke-Pohst.

(These should address both items in blue on the previous slide.)

Smart/Wildanger Decompositions

- Define certain sets of solutions:

$$L := \{\tau \in \boldsymbol{\Gamma} : \tau_0 + \tau_1 = 1\}$$
$$L_H := \{\tau \in L : \tau_i = \rho^{\mathbf{b}_i},\ |\mathbf{b}_i| \leq H\}$$
$$L_H(R) := \{\tau \in L_H : R^{-1} \leq |\alpha|_{\mathfrak{p}} \leq R, \forall\, \mathfrak{p} \in S_{\text{fin}}\}$$

- There are large H, R such that $L = L_H(R)$.

- For $H' \leq H$ and $R' \leq R$,

$$L_H(R) = L_{H'}(R') \cup \bigcup_{j=1}^{4} \bigcup_{\mathfrak{p}\in S} T_{j,\mathfrak{p},H,R,R'}$$

where elements of $T_{i,\mathfrak{p}} := T_{i,\mathfrak{p},H,R,R'}$ are "extreme" with respect to $\mathfrak{p}$ or some exponent on ρ.

Smart/Wildanger Decompositions

$$L_H(R) = L_{H'}(R') \cup \bigcup_{j=1}^{4} \bigcup_{\mathfrak{p} \in S} T_{j,\mathfrak{p}}$$

- "LLL-type" arguments allow us to argue $T_{j,\mathfrak{p}} = \varnothing$. This leads to improvements on each exponent bound $K_1(\ell)$.
- Nonempty? Solutions in $T_{j,\mathfrak{p}}$ still correspond to vectors which ...
  - belong to a lattice generated by an explicit matrix, A, and
  - belong to a "small" ellipsoid.
- The algorithm of Fincke-Pohst can search for all such lattice points.
- Once H and R are sufficiently small, Fincke-Pohst can also search for solutions inside $L_H(R)$!

Other To-Dos

- Allow $\alpha \neq (1,1)$, or $\Gamma_i \neq \mathcal{O}_{K,S}^{\times}$.
- Allow Galois constraints.
- Decompose search into disjoint pieces (for parallelization/pausing).
- Python to Cython where possible.

S-Unit Community

With assistance from:

Norman Danner, Bjorn Poonen, David Roe, Andrew Sutherland, . . .

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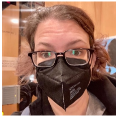
A. Alvarado

A. Koutsianas

B. Malmskog

C. Rasmussen



C. Vincent

M. West

And building on the past work of:

Baker, Baker-Wüstholz, Bremner, Brumer, de Weger, Evertse-Győry, Gelfond, Győry, Győry-Yu, Koutsianas, Lenstra-Lenstra Lovász, Mahler, Malmskog-R., Merriman-Smart, Pethö-de Weger, SageMath Developers, Schneider, Siegel, Smart, Tzanakis-de Weger, Wildanger, Yu, . . .