On the growth of recurrences

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Quotation

A.J. van der Poorten in *Some facts that should be better known, especially about rational functions*:

Recurrence sequences [...] are popular amongst professionals and amateurs alike. Yet it is peculiar difficult to find congenial summaries of well known basic facts, whilst recent deep results remain hidden in the technical literature. [...] But much of what I write is influenced [...] by my being reminded, all too frequently, that the well known is often not generally known, let alone known well.

Recurrences and multi-recurrences

A multi-recurrence is a map $G : \mathbb{N}^s \to K$ such that

$$G(n_1, \ldots, n_s) = \sum_{i=1}^{k} f_i(n_1, \ldots, n_s) \alpha_{i1}^{n_1} \cdots \alpha_{is}^{n_s}$$

where s and k are positive integers, $f_1, \ldots, f_k$ are polynomials in s variables and $n_1, \ldots, n_s$ are non-negative integers. We say that G is defined over a field K if the coefficients and the bases $\alpha_{i1}, \ldots, \alpha_{is}$ for $i = 1, \ldots, k$ are in K. If G is defined over K, then it takes values in K.

If $s = 1$, then the multi-recurrence is a recurrence; we then write $G_n = G(n)$. A recurrence is called non-degenerate if no quotient of two distinct characteristic roots is a root of unity resp. a non-zero constant and it is called simple if all characteristic roots are simple roots.

Growth of recurrences

Let $(G_n)_{n=0}^{\infty}$ be a non-degenerate linear recurrence sequence taking values in a number field and let $G_n = f_1(n)\alpha_1^n + \cdots + f_t(n)\alpha_t^n$ with algebraic integers $\alpha_1, \ldots, \alpha_t$ be its power sum representation satisfying $|\alpha_1| > |\alpha_2| \geq \cdots \geq |\alpha_t|$ and $|\alpha_1| > 1$.

Then for any $\varepsilon > 0$ we have

$$\begin{aligned} |G_n| &= |f_1(n)||\alpha_1|^n \left| 1 + \frac{f_2(n)\alpha_2^n + \cdots + f_t(n)\alpha_t^n}{f_1(n)\alpha_1^n} \right| \\ &\geq |f_1(n)||\alpha_1|^n \left| 1 - \left| \frac{f_2(n)\alpha_2^n + \cdots + f_t(n)\alpha_t^n}{f_1(n)\alpha_1^n} \right| \right| \\ &\geq \frac{1}{2}|f_1(n)||\alpha_1|^n \\ &\geq |\alpha_1|^{n(1-\varepsilon)} \end{aligned}$$

for n sufficiently large.

Growth of recurrences. II

Theorem (van der Poorten)

Let $(G_n)_{n=0}^{\infty}$ be a non-degenerate linear recurrence sequence taking values in a number field K and let $G_n = f_1(n)\alpha_1^n + \cdots + f_t(n)\alpha_t^n$ with algebraic integers $\alpha_1, \ldots, \alpha_t$ be its power sum representation satisfying $\max_{j=1,\ldots,t} |\alpha_j| > 1$. Then for any $\varepsilon > 0$ the inequality

$$|G_n| \geq \left(\max_{j=1,\ldots,t} |\alpha_j|\right)^{n(1-\varepsilon)}$$

is satisfied for every n sufficiently large.

Growth of recurrences. III

Theorem (Akiyama-Evertse-Pethő)

Let $r \geq 2$ be an integer and $h > 1$ be a real number. There exists a linear recurrence sequence $(G_n)_{n=0}^{\infty}$ of order r such that $G_n \neq 0$ for all n, $|G_n| \gg h^n$ for infinitely many n and $|G_n| \ll h^{-n}$ for infinitely many n.

5.2.5 *A Confession.* It was this question that moved me to concoct utterly fallacious proofs (please, do not look at [29] and [73]) of its answer and other celebrated conjectures of the subject. En route, I needed a growth estimate for recurrence sequences and stumbled upon the much deeper results mentioned at 4.3 and applied above. For [29] there is a corrigendum [71] of sorts (some claims are still too sloppy), but [73] defies repair since some of its 'results' cannot be true. The criterion at 4.4.5 is at 4.4.5 is a sanitised version of the viciously false allegation at [73], 1284-85. Those claims which I have managed to retrieve, necessarily by different arguments from those originally suggested, are mention below at 6.

Growth of multi-recurrences

Theorem (van der Poorten-Schlickewei)

Let K be a number field and s a positive integer. Consider the polynomial-exponential function

$$G(\mathbf{n}) = \sum_{i=1}^{k} f_i(\mathbf{n})\boldsymbol{\alpha}_i^{\mathbf{n}}$$

with non-zero algebraic integers $\alpha_{ij} \in K$, for $i = 1, \ldots, k$ and $j = 1, \ldots, s$, and polynomials $f_i(X_1, \ldots, X_s) \in K[X_1, \ldots, X_s]$. Fix $\varepsilon > 0$. Assume that there is an index i_0, $1 \le i_0 \le k$, such that there is no subset $I \subseteq \{1, \ldots, k\}$ with $i_0 \in I$ and

$$\sum_{i \in I} f_i(\mathbf{n})\boldsymbol{\alpha}_i^{\mathbf{n}} = 0.$$

...

Growth of multi-recurrences. II

Theorem (van der Poorten-Schlickewei)

Then, for $|\mathbf{n}|$ large enough we have

$$|G(\mathbf{n})| \geq \left| f_{i_0}(\mathbf{n}) \alpha_{i_0}^{\mathbf{n}} \right| e^{-\varepsilon |\mathbf{n}|}.$$

Growth of multi-recurrences. III

- The condition concerning i_0 in the above theorem is really necessary and already stated by van der Poorten and Schlickewei in 1982. Indeed, the size of $G(\mathbf{n})$ cannot be bounded by a term from a vanishing subsum.
- The same statement holds with the completely analogous proof also for any other valuation $|\cdot|_\mu$ on K in the proven lower bound instead of the standard absolute value.
- The bound holds for all $\mathbf{n}$ with $|\mathbf{n}| \geq B$. Unfortunately this lower bound B cannot be given explicitly since it depends, among others, on the ineffective constant given Evertse's Theorem below. More precisely, it is influenced by a threshold where the exponential function becomes larger than a polynomial function having ineffective coefficients.

Tool

For the tool used we need some notation. Let

$$\|\mathbf{x}\| := \max_{\substack{k=0,\dots,t \\ i=1,\dots,D}} |\sigma_i(x_k)|$$

with $\{\sigma_1, \dots, \sigma_D\}$ the set of all embedding of K in $\mathbb{C}$ and $\mathbf{x} = (x_0, x_1, \dots, x_t)$. Moreover, we denote by $\mathcal{O}_K$ the ring of integers in the number field K and by M_K the set of places of the number field K.

Tool. II

Theorem (Evertse)

Let t be a non-negative integer and S a finite set of places in K, containing all infinite places. Then for every $\varepsilon > 0$ a constant C exists, depending only on ε, S, K, t such that for each non-empty subset T of S and every vector $\mathbf{x} = (x_0, x_1, \ldots, x_t) \in \mathcal{O}_K^{t+1}$ with

$$x_{i_0} + x_{i_1} + \cdots + x_{i_s} \neq 0$$

for each non-empty subset $\{i_0, i_1, \ldots, i_s\}$ of $\{0, 1, \ldots, t\}$ the inequality

$$\left(\prod_{k=0}^{t} \prod_{\nu \in S} |x_k|_\nu\right) \prod_{\nu \in T} |x_0 + \cdots + x_t|_\nu \geq C \left(\prod_{\nu \in T} \max_{k=0,\ldots,t} |x_k|_\nu\right) \|\mathbf{x}\|^{-\varepsilon}$$

is valid.

Proof (after F.-Heintze)

We can find a non-zero integer z such that $zf_i(\mathbf{n})\boldsymbol{\alpha}_i^{\mathbf{n}}$ are algebraic integers for all $i = 1, \ldots, k$ and all non-negative integers $n_1, \ldots, n_s$.

Choose S as a finite set of places in K containing all infinite places as well as all places such that α_{ij} for $i = 1, \ldots, k$ and $j = 1, \ldots, s$ are S-units.

Let μ be such that $|\cdot|_\mu = |\cdot|$ is the usual absolute value on $\mathbb{C}$. In particular we have $\mu \in S$. Further define $T = \{\mu\}$.

We may assume that for the index i_0 from the theorem we have $i_0 = 1$ to simplify the notation. By renumbering summands we can assume that

$$G(\mathbf{n}) = \sum_{i=1}^{\ell} f_i(\mathbf{n})\boldsymbol{\alpha}_i^{\mathbf{n}}$$

for an integer ℓ with $1 \leq \ell \leq k$ has no vanishing subsum.

Proof (after F.-Heintze). II

We aim to apply Evertse's theorem. We have

$$\left(\prod_{i=1}^{\ell}\prod_{\nu\in S}|zf_i(\mathbf{n})\boldsymbol{\alpha}_i^{\mathbf{n}}|_\nu\right)|zG(\mathbf{n})| \geq C \max_{i=1,\ldots,\ell}|zf_i(\mathbf{n})\boldsymbol{\alpha}_i^{\mathbf{n}}|\,\|z\mathbf{x}\|^{-\varepsilon'}$$

for $\mathbf{x}=(f_1(\mathbf{n})\boldsymbol{\alpha}_1^{\mathbf{n}},\ldots,f_\ell(\mathbf{n})\boldsymbol{\alpha}_\ell^{\mathbf{n}})$ and an ε' to be fixed later. Using that z is a fixed integer and that the α_{ij} are S-units, we get

$$\left(\prod_{i=1}^{\ell}\prod_{\nu\in S}|f_i(\mathbf{n})|_\nu\right)|G(\mathbf{n})| \geq C_1\,|f_1(\mathbf{n})\boldsymbol{\alpha}_1^{\mathbf{n}}|\,\|\mathbf{x}\|^{-\varepsilon'}.$$

The part with the polynomials is easy to bound by

$$\prod_{i=1}^{\ell}\prod_{\nu\in S}|f_i(\mathbf{n})|_\nu \leq \prod_{i=1}^{\ell}C_2^{(i)}\,|\mathbf{n}|^{Dm} \leq C_3\,|\mathbf{n}|^{Dm\ell},$$

where m denotes the maximum of the absolute degrees of the polynomials $f_1,\ldots,f_k$ and D is the degree of K.

Proof (after F.-Heintze). III

There exists a constant $A > 1$, which is independent of $\mathbf{n}$ and ε' satisfying

$$\begin{aligned}
\|\mathbf{x}\| &= \max_{\substack{i=1,\dots,\ell \\ t=1,\dots,D}} \left|\sigma_t\left(f_i(\mathbf{n})\boldsymbol{\alpha}_i^{\mathbf{n}}\right)\right| \\
&\leq \max_{\substack{i=1,\dots,\ell \\ t=1,\dots,D}} \left|\sigma_t\left(f_i(\mathbf{n})\right)\right| \cdot \max_{\substack{i=1,\dots,\ell \\ t=1,\dots,D}} \left|\sigma_t\left(\boldsymbol{\alpha}_i^{\mathbf{n}}\right)\right| \\
&\leq C_4 \left|\mathbf{n}\right|^m \prod_{j=1}^{s} \max_{\substack{i=1,\dots,\ell \\ t=1,\dots,D}} \left|\sigma_t\left(\alpha_{ij}^{n_j}\right)\right| \\
&\leq C_4 \left|\mathbf{n}\right|^m A^{|\mathbf{n}|}.
\end{aligned}$$

Proof (after F.-Heintze). IV

We get

$$C_3 \, |\mathbf{n}|^{Dm\ell} \, |G(\mathbf{n})| \geq |f_1(\mathbf{n})\boldsymbol{\alpha}_1^{\mathbf{n}}| \, C_5 \, |\mathbf{n}|^{-m\varepsilon'} \, A^{-|\mathbf{n}|\varepsilon'}.$$

Thus

$$\begin{aligned} |G(\mathbf{n})| &\geq |f_1(\mathbf{n})\boldsymbol{\alpha}_1^{\mathbf{n}}| \, C_6 \, |\mathbf{n}|^{-Dm\ell - m\varepsilon'} \, A^{-|\mathbf{n}|\varepsilon'} \\ &\geq |f_1(\mathbf{n})\boldsymbol{\alpha}_1^{\mathbf{n}}| \, A^{-2\varepsilon'|\mathbf{n}|} \end{aligned}$$

where the last inequality holds for $|\mathbf{n}|$ large enough. Thus, choosing ε' such that $2\varepsilon' \log(A) = \varepsilon$, we end up with

$$|G(\mathbf{n})| \geq |f_1(\mathbf{n})\boldsymbol{\alpha}_1^{\mathbf{n}}| \, e^{-2\varepsilon' \log(A)|\mathbf{n}|} = |f_1(\mathbf{n})\boldsymbol{\alpha}_1^{\mathbf{n}}| \, e^{-\varepsilon|\mathbf{n}|}$$

and the theorem is proven.

Growth of recurrences. II

Theorem (van der Poorten)

Let $(G_n)_{n=0}^{\infty}$ be a non-degenerate linear recurrence sequence taking values in a number field K and let $G_n = f_1(n)\alpha_1^n + \cdots + f_t(n)\alpha_t^n$ with algebraic integers $\alpha_1, \ldots, \alpha_t$ be its power sum representation satisfying $\max_{j=1,\ldots,t} |\alpha_j| > 1$. Then for any $\varepsilon > 0$ the inequality

$$|G_n| \geq \left(\max_{j=1,\ldots,t} |\alpha_j| \right)^{n(1-\varepsilon)}$$

is satisfied for every n sufficiently large.

Growth of recurrences in function fields

Theorem (F.-Heintze)

Let $(G_n)_{n=0}^{\infty}$ *be a non-degenerate linear recurrence sequence taking values in a function field* K *in one variable over* $\mathbb{C}$ *with power sum representation* $G_n = a_1(n)\alpha_1^n + \cdots + a_t(n)\alpha_t^n$*. Let* L *be the splitting field of the characteristic polynomial of that sequence, i.e.* $L = K(\alpha_1, \ldots, \alpha_t)$*. Moreover, let* μ *be a valuation on* L*. Then there is an effectively computable constant* C*, independent of* n*, such that for every sufficiently large* n *the inequality*

$$\mu(G_n) \leq C + n \cdot \min_{j=1,\ldots,t} \mu(\alpha_j)$$

holds.

Growth of recurrences in function fields. II

- The reverse inequality is trivially valid (with another constant).
- If the G_n are polynomials with polynomial characteristic roots and if $\mu = \deg$, then the bound takes the form

$$\deg G_n \geq n \cdot \max_{i=1,\dots,j} \deg \alpha_i - C.$$

- The result answers a question of van der Poorten-Shparlinski in the situation we are working in.

Tools

Theorem (Zannier)

Let $F/\mathbb{C}$ be a function field in one variable, of genus $\mathfrak{g}$, let $\varphi_1, \dots, \varphi_n \in F$ be linearly independent over $\mathbb{C}$ and let $r \in \{0, 1, \dots, n\}$. Let S be a finite set of places of F containing all the poles of $\varphi_1, \dots, \varphi_n$ and all the zeros of $\varphi_1, \dots, \varphi_r$. Put $\sigma = \sum_{i=1}^n \varphi_i$. Then

$$\sum_{\nu \in S} \left(\nu(\sigma) - \min_{i=1,\dots,n} \nu(\varphi_i) \right) \leq \binom{n}{2} (|S| + 2\mathfrak{g} - 2) + \sum_{i=r+1}^{n} \mathcal{H}(\varphi_i).$$

Theorem (F.-Pethő)

Let K be as above and L be a finite extension of K of genus $\mathfrak{g}$. Let further $\alpha_1, \ldots, \alpha_d \in L^*$ with $d \geq 2$ be such that $\alpha_i/\alpha_j \notin \mathbb{C}^*$ for each pair of subscripts i, j with $1 \leq i < j \leq d$. Moreover, for every $i = 1, \ldots, d$ let $\pi_{i1}, \ldots, \pi_{ir_i} \in L$ be r_i linearly independent elements over $\mathbb{C}$. Put

$$q = \sum_{i=1}^{d} r_i.$$

Then for every $n \in \mathbb{N}$ such that $\{\pi_{il}\alpha_i^n : l = 1, \ldots, r_i, i = 1, \ldots, d\}$ is linearly dependent over $\mathbb{C}$, but no proper subset of this set is linearly dependent over $\mathbb{C}$, we have

$$n \leq C = C(q, \mathfrak{g}, \pi_{il}, \alpha_i : l = 1, \ldots, r_i, i = 1, \ldots, d).$$

Sketch of the proof

Denote the coefficients of the polynomial $a_j(n) \in L[n]$ by $a_{j0}, a_{j1}, \ldots, a_{jm_j}$ where m_j is the degree of $a_j(n)$. So

$$a_j(n) = \sum_{k=0}^{m_j} a_{jk} n^k.$$

We may assume that $t \geq 2$. Let $\pi_{j1}, \ldots, \pi_{jk_j}$ be a maximal $\mathbb{C}$-linear independent subset of $a_{j0}, a_{j1}, \ldots, a_{jm_j}$. Then we can write the sequence in the form

$$G_n = \sum_{j=1}^{t} \left(\sum_{i=1}^{k_j} b_{ji}(n) \pi_{ji} \right) \alpha_j^n$$

with complex polynomials $b_{ji}(n)$. We may assume that $b_{ji} \neq 0$. Moreover we assume that n is large enough such that $b_{ji}(n) \neq 0$ for all j, i.

Consider the set $M := \{\pi_{ji}\alpha_j^n : i = 1, \dots, k_j, j = 1, \dots, t\}$. If M is linearly dependent over $\mathbb{C}$, then we choose a minimal linear dependent subset $\widetilde{M}$ of M, i.e. a linearly dependent subset $\widetilde{M}$ with the property that no proper subset of $\widetilde{M}$ is linearly dependent. Let $\widetilde{G_n}$ be the linear recurrence sequence associated with this subset $\widetilde{M}$, that is

$$\widetilde{G_n} = \sum_{j=1}^{s} \left(\sum_{i=1}^{\widetilde{k_j}} b_{ji}(n)\pi_{ji} \right) \alpha_j^n$$

for $s \leq t$ and after a suitable renumbering of the summands. Since $\pi_{j1}, \dots, \pi_{jk_j}$ are $\mathbb{C}$-linear independent we have $s \geq 2$. Applying F.-Pethő to

$$\widetilde{M} := \left\{ \pi_{ji}\alpha_j^n : i = 1, \dots, \widetilde{k_j}, j = 1, \dots, s \right\}$$

gives an upper bound for n. Thus for n large enough M cannot be linearly dependent.

Sketch of the proof. III

Since for each fixed n we have $b_{ji}(n) \in \mathbb{C}^*$, the set

$$M' := \left\{ b_{ji}(n)\pi_{ji}\alpha_j^n : i = 1, \ldots, k_j, j = 1, \ldots, t \right\}.$$

is linearly independent over $\mathbb{C}$ and contains for each $j = 1, \ldots, t$ at least one element.

Let S be a finite set of places of L containing all zeros and poles of α_j for $j = 1, \ldots, t$ and of the nonzero a_{ji} for $j = 1, \ldots, t$ and $i = 1, \ldots, m_j$ as well as μ and the places lying over ∞. Put $r = |M'|$ and let the φ_i's be the elements of M'. Applying Zannier's theorem yields

$$\sum_{\nu \in S} \left(\nu(G_n) - \min_{\substack{j=1,\ldots,t \\ i=1,\ldots,k_j}} \nu\left(b_{ji}(n)\pi_{ji}\alpha_j^n\right) \right) \le \binom{\sum_{j=1}^t k_j}{2} (|S| + 2\mathfrak{g} - 2),$$

where the bound on the right will be denoted by C_1.

Sketch of the proof. IV

Since each summand in the sum on the left hand side is non-negative

$$\mu(G_n) - \min_{\substack{j=1,\ldots,t \\ i=1,\ldots,k_j}} \mu\left(b_{ji}(n)\pi_{ji}\alpha_j^n\right) \leq C_1.$$

Therefore for all $j_0 = 1, \ldots, t$ and $i_0 = 1, \ldots, k_{j_0}$ we get

$$\begin{aligned} \mu(G_n) &\leq C_1 + \min_{\substack{j=1,\ldots,t \\ i=1,\ldots,k_j}} \mu\left(b_{ji}(n)\pi_{ji}\alpha_j^n\right) \\ &\leq C_1 + \mu\left(b_{j_0 i_0}(n)\pi_{j_0 i_0}\alpha_{j_0}^n\right) \\ &= C_1 + \mu\left(\pi_{j_0 i_0}\right) + n\mu\left(\alpha_{j_0}\right) \\ &= C_2 + n\mu\left(\alpha_{j_0}\right). \end{aligned}$$

Since this holds for all $j_0 = 1, \ldots, t$ we have

$$\mu(G_n) \leq C_2 + n \cdot \min_{j=1,\ldots,t} \mu(\alpha_j).$$

Some open problems

- Find and prove an analogous result for the growth of multi-recurrences in the function field case.
- Find and prove an analogous result for the growth of recurrences without algebraic coefficients in the function field case.

References

- A. J. van der Poorten, Some problems of recurrent interest, Topics in classical number theory, Vol. I, II (Budapest, 1981), 1265-1294, *Colloq. Math. Soc. Janos Bolyai* **34**, North-Holland, Amsterdam, 1984.
- A. J. van der Poorten, Some facts that should be better known, especially about rational functions, In *Number Theory and Applications*, Kluwer Academic, 1989, 497-528.
- A. J. van der Poorten and H. P. Schlickewei, The growth conditions for recurrence sequences, *Macquarie Math. Reports* **82-0041** (August 1982), Macquarie University, Australia, 2109.
- A. J. van der Poorten and I. Shparlinski, On sequences of polynomials defined by certain recurrence relations, *Acta Sci. Math. (Szeged)* **61** (1995), 77-103.

References (cont.)

- S. Akiyama, J.-H. Evertse and A. Pethő, On nearly linear recurrence sequences, Number theory - Diophantine problems, uniform distribution and applications, 1-24, Springer, Cham, 2017.
- J.-H. Evertse, On sums of S-units and linear recurrences, *Comp. Math.* **53** (1984), no. 2, 225-244.
- C. Fuchs and S. Heintze, On the growth of linear recurrences in function fields, *Bull. Austr. Math. Soc.* **104(1)** (2021), 11-20.
- C. Fuchs and S. Heintze, On the growth of multi-recurrences, *Arch. Math. (Basel)*, to appear.
- C. Fuchs and A. Pethő, Effective bounds for the zeros of linear recurrences in function fields, *J. Theor. Nombres Bordeaux* **17** (2005), 749-766.
- U. Zannier, On composite lacunary polynomials and the proof of a conjecture of Schinzel, *Invent. Math.* **174** (2008), no. 1, 127-138.

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