Power values of pyramidal numbers

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Prologue - Play with figurate numbers

Generalized pyramidal numbers:

$$S_m^k(n) = \frac{n(n+1)\cdots(n+k-2)((m-2)n-m+k+2)}{k!}$$

Some important special cases: Binomial coefficients

$$S_3^k(n) = \frac{n(n+1)\cdots(n+k-2)(n+k-1)}{k!}$$

Pyramidal numbers

$$S_m^3(n) = \frac{n(n+1)((m-2)n-m+5)}{6} = \mathrm{Pyr}_m(n)$$

$$S_3^3(n) = \frac{n(n+1)(n+2)}{6} = \binom{n+2}{3} = \mathrm{Pyr}_3(n)$$

$$S_4^3(n) = \frac{n(n+1)(2n+1)}{6} = 1^2 + 2^2 + \ldots + n^2 = \mathrm{Pyr}_4(n)$$

Two earlier diophantine results

On the power values of binomial coefficients: Győry, Acta Aritmetica, 1996

On the power values of power sums: Bennett, Győry, P, Compositio, 2004

Corollary

All the solutions of the equations

$$\binom{x}{3} = y^n \quad \text{and} \quad 1^2 + 2^2 + \ldots + x^2 = y^n$$

in integers $x \geq 3, y > 1$ and $n \geq 2$ are

$$(x, y, n) = (50, 140, 2) \quad \text{and} \quad (x, y, n) = (24, 70, 2),$$

respectively.

Let m be a fixed integer with $m \geq 3$ and denote by

$$\mathrm{Pyr}_m(x) = \frac{1}{6}x(x+1)((m-2)x+5-m)$$

the mth order pyramidal number.

In general, we consider the equation

$$\mathrm{Pyr}_m(x) = \frac{1}{6}x(x+1)((m-2)x+5-m) = y^n$$

in integers x, y, m and n with $x \geq 1, y > 1, m \geq 3$ and $n \geq 2$.

$n = 2$, Dujella, Győry, P, 2012, Acta Arithmetica

For $n = 2$, the problem is equivalent to finding the integer points on the elliptic curve

$$E_m : y^2 = \frac{1}{6}x(x+1)((m-2)x+5-m)$$

with $m \geq 3, m \neq 5$.

Trivial integral points: $(0,0), (-1,0), (1,1)$ and for $m \equiv 2 \mod 3$

$$\left(\frac{2m-10}{3}, \frac{(m-5)(2m-7)}{9}\right).$$

Non-trivial solutions I

Apart from the trivial points, in the range $3 \leq m \leq 100, m \neq 5$, all integer points (m, x, y) on E_m are:

$$(3, -2, 0), (3, 2, 2), (3, 48, 140), (4, 24, 70), (7, 6, 14), (7, 49, 315),$$

$$(11, 1681, 84419), (13, 24, 160), \underline{(15, 2, 4)}, (15, 242, 5544),$$

$$(16, 49, 525), (20, 49, 595), (24, 2, 5), (24, 1681, 131979),$$

$$(28, 23, 230), (29, 8, 48), (33, 7, 42), (35, 2, 6), (35, 49, 805),$$

$$(41, 4, 20), (41, 49, 875), (45, 120, 3520), (48, 2, 7), (52, 96, 2716),$$

$$(53, 1681, 200941), (62, 49, 1085), (63, 2, 8), (63, 16, 204),$$

$$(68, 24, 390), (68, 343, 21070), (68, 57121, 45278311), \underline{(70, 6, 49)},$$

Non-trivial solutions II

$$(70,49,1155),(73,833,82705),(74,8,78),(75,10,110),$$
$$(76,289,17255),(76,3479,720650),\underline{(80,2,9)},(80,1681,248501),$$
$$(89,7,70),(91,4,30),(91,6,56),(97,49,1365),(98,8,90),$$
$$(99,2,10),(99,57121,54891369).$$

Observation: $(m, x, y) = (15, 2, 2^2)$ and $(m, x, y) = (17, 8, 6^2)$ for $m \leq 50$

The proof is based on the subroutine IntegralPoints of the program package MAGMA and the subroutine integral points of the program package SAGE.

This result is from 2012. Do you extend the resolution for larger m by using up-to-date hardware and software?

$n \geq 3$, Dujella, Győry, Michaud-Jacobs, and P, Acta Arithmetica, submitted

For $n \geq 3$ we have the following result:

All the solutions of the equation

$$\mathrm{Pyr}_m(x) = \frac{1}{6}x(x+1)((m-2)x+5-m) = y^n$$

in integers x, y, m and n with $x \geq 1, y > 1, 3 \leq m \leq 50$ and $n \geq 3$ are

$$(m, x, y, n) = (5, 57121, 3107, 4), (7, 2, 2, 3), (15, 2, 2, 4), (17, 8, 6, 4),$$

$$(26, 2, 3, 3), (31, 2, 2, 5) \quad \text{and} \quad (50, 15, 30, 3).$$

n=4

$$\mathrm{Pyr}_m(x) = \frac{1}{6}x(x+1)((m-2)x+5-m) = y^4 = z^2$$

Using our observation we have two solutions

$$(m, x, y) = (15, 2, 2) \quad \text{and} \quad (17, 8, 6)$$

m=5

Let $m = 5$ and (x, y, n) be a solution to the equation

$$\frac{x^2(x+1)}{2} = y^n$$

with $x > 0, y > 1$, and $n \geq 3$. Then

$$(x, y, n) = (57121, 3107, 4).$$

$n = 4$

We have to distinguish two cases: $n = 4$ or $n \geq 3$ is odd.

For $n = 4$ we obtain

$$x^2(x+1) = 2y^4.$$

One can see that x is odd, so

$$x^2\left(\frac{x+1}{2}\right) = y^4,$$

and $\gcd(x^2, (x+1)/2) = 1$.

It follows that $x^2 = y_1^4$ and $x + 1 = 2y_2^4$, and

$$y_1^2 + 1 = 2y_2^4.$$

This is Ljunggren's equation. It has two positive integer solutions

$$(y_1, y_2) = (1, 1), (239, 13).$$

Since $y = y_1 y_2 > 1$ we get the unique solution

$$(x, y) = (y_1^2, y_1 y_2) = (57121, 3107).$$

n is odd

By an elementary manipulation we have binomial Thue-like equation

$$ax^n - by^n = 1,$$

with $ab = 2^\alpha$, where α is a positive integer.

From a known result of Bennett (see Bulletin of London Math. Society, Products of consecutive integers, 2004) there is only trivial solution $xy = 1$ to the Thue equations for every n.

A trivial and important observation

The polynomial

$$x(x+1)((m-2)x+5-m)$$

has three rational factors.

In the sequel we suppose that $n = p \geq 3$ is a prime, and that $6 \leq m \leq 50$

Our problem is to resolve the equation

$$\mathrm{Pyr}_m(x) = \frac{1}{6}x(x+1)((m-2)x+5-m) = y^p.$$

We rewrite in the following form

$$x(x+1)(a_m x - b_m) = c_m y^p,$$

where

$$(a_m, b_m, c_m) = \begin{cases} \left(\frac{m-2}{3}, \frac{m-5}{3}, 2\right) & \text{if } \ m \equiv 2 \pmod 3, \\ (m-2, m-5, 6) & \text{if } \ m \not\equiv 2 \pmod 3. \end{cases}$$

Simultaneous Thue equations

The previous equation leads us to consider the following system of binomial Thue equations

$$By_2^p - Ay_1^p = 1$$
$$a_m Ay_1^p - Cy_3^p = b_m.$$

Here (A, B, C) belongs to a finitely and effectively determinable set of triples depending on m.

Upper bound for p

Let (x, y, m, p) be a solution of the equation

$$x(x+1)(a_m x - b_m) = c_m y^p$$

with $y > 1$. Then

$$p < 10676 \cdot \log\left(c_m^2 \cdot b_m \cdot (a_m + b_m)\right).$$

The proof is based on a Baker-type inequality developed by Mignotte in an Acta Arithmetica's paper A note on the equation $ax^n - by^n = c$ from 1996.

Determination of the set of (A, B, C)

We will distinguish six cases according to $\mathrm{ord}_2(b_m) = 0, 1, 2, 3, 4, 5$

Example: $\mathrm{ord}_2(b_m) = 1$

In this case

$$m \in \{7, 11, 15, 19, 23, 27, 31, 35, 39, 47\}.$$

We have

- $b_m = 2 \cdot {p_1}^{r_1}$ for some $r_1 \in \{0, 1\}$, and $p_1 \nmid c_m$ is prime;
- $a_m + b_m = p_2 \cdot {q_2}^{s_2}$, for some $s_2 \in \{0, 1\}$, and $p_2, q_2 \nmid c_m$ are prime.

Then

$$A = 2^{\alpha_1} \cdot 3_1^{\beta} \cdot {p_1}^{\gamma_1},$$

$$B = 2^{\alpha_2} \cdot 3^{\beta_2} \cdot {p_2}^{\gamma_2} \cdot {q_2}^{\delta_2},$$

$$C = 2^{\alpha_3} \cdot 3^{\beta_3} \cdot {p_1}^{p-\gamma_1} \cdot {p_2}^{p-\gamma_2} \cdot {q_2}^{p-\delta_2},$$

where

$$(\alpha_1, \alpha_2, \alpha_3) \in \{(1,0,0),(0,1,0),(0,0,1)\},$$

$$(\beta_1, \beta_2, \beta_3) \in \begin{cases} \{(1,0,0),(0,1,0),(0,0,1)\} & \text{if } c_m = 6, \\ \{(0,0,0)\} & \text{if } c_m = 2, \end{cases}$$

$$\gamma_1 \in \{0, r_1, p - r_1\},$$

$$\gamma_2 \in \{0, 1, p - 1\},$$

$$\delta_2 \in \{0, s_2, p - s_2\}.$$

We consider the system of Thue equations

$$\begin{aligned} B\,y_2^p - A\,y_1^p &= 1 \\ a_m A\,y_1^p - C\,y_3^p &= b_m. \end{aligned}$$

for a fixed value of m and triple (A, B, C). We will start by considering this system mod ℓ, for many auxiliary primes ℓ to try and obtain a contradiction; since if the system of equations has no local solution then it will certainly not have a global solution. When the system of equations does not have a (global) solution, we found this method to be extremely effective (as we see below).

The algorithm

Fix a prime $p > 2$. We search for a prime ℓ such that $\ell = 2kp + 1$ for some $k \geq 1$ (i.e. $\ell \equiv 1 \pmod{p}$), such that $\ell \nmid ABC$, and for which the system of equations has no solution mod ℓ. If we can find such an ℓ, then we have obtained a contradiction. The reason for choosing ℓ of this form is that we have, for each $i \in \{1, 2, 3\}$, either $\ell \mid y_i$, or

$$({y_i}^p)^{2k} = {y_i}^{\ell-1} \equiv 1 \pmod{\ell}.$$

In particular, ${y_i}^p \in \mu_{2k}(\mathbb{F}_\ell) \cup \{0\}$, where $\mu_{2k}(\mathbb{F}_\ell) = \{\alpha \in \mathbb{F}_\ell : \alpha^{2k} = 1\}$. We therefore only have $2k + 1$ possibilities for ${y_i}^p \pmod{\ell}$, and moreover the set $\mu_{2k}(\mathbb{F}_\ell)$ can be computed extremely quickly using a primitive root modulo ℓ. Indeed, if g is a primitive root modulo ℓ, then

$$\mu_{2k}(\mathbb{F}_\ell) = \{(g^p)^r : 0 \leq r \leq 2k - 1\}.$$

For each triple (A, B, C), we searched for a prime ℓ by testing with $1 \leq k \leq 150$.

For $p > 5$, with p less than the bound

$$10676 \cdot \log\left(c_m^2 \cdot b_m \cdot (a_m + b_m)\right),$$

apart from the cases where we have a global solution, and a single case when $p = 7$, we succeeded in obtaining a contradiction.

Example

$$3 \cdot x^5 - 2 \cdot y^5 = c$$
$$3 \cdot x^5 - 2 \cdot y^5 \equiv c \pmod{11}$$
$$x^5, y^5 \in \{0, 1, 10\} \pmod{11}$$
$$3 \cdot x^5 - 2 \cdot y^5 \in \{0, 1, 2, 3, 5, 6, 8, 9, 10\}$$

If $c \equiv 4$ or $7 \pmod{11}$, then our binomial Thue equation has no solution.

When $p = 3$ or $p = 5$, the method sometimes fails even when there is no global solution. In these cases, as p is small we can simply solve the two Thue equations using `Magma` and verify whether we have a solution (y_1, y_2, y_3) with $y_1, y_2 > 0$ (since $x > 0$). As mentioned above, the local method also fails for $p = 7$ in a single case. This is for the case $m = 21$ and $(A, B) = (2^4 \cdot 3, 1)$. Here we also simply solve the corresponding Thue equations directly to conclude there are no non-zero solutions.

Local method fails

(I) $A = 1, B = 2$, and $a_m - C = b_m$. Here we have a global solution $(y_1, y_2, y_3) = (1, 1, 1)$ for all p, which comes from the solution $x = y = 1$ to our original equation. However, in this case, our first Thue equation is

$$2{y_2}^p - {y_1}^p = 1.$$

Applying a well-known result of Bennett (Rational approximation to algebraic numbers of small height: the Diophantine equation $|ax^n + by^n| = 1$, appeared in Crelle's Journal, 2001), we see that $y_1 = y_2 = 1$ for all p, so $x = 1$.

(II) $A = 1$ and $C = a_m + b_m$. This admits the solution $(y_1, y_2, y_3) = (-1, 0, -1)$.

(III) $B = 1$ and $C = b_m$. This admits the solution $(y_1, y_2, y_3) = (0, 1, -1)$.

Modular method

It remains to deal with cases (II) and (III), outlined in Section 4, for each $6 \leq m \leq 50$. In each case, we have $A = 1$ or $B = 1$, and this leads to an equation of the form

$$z_1^p - Dz_2^p = 1$$

for integers z_1 and z_2. The following result of Bartolomé and Mihăilescu (International Journal Of Number Theory, 2017) will be extremely helpful.

Let $D > 1$ and and let p be an odd prime satisfying

$$\gcd(\mathrm{Rad}(\varphi(D)), p) = 1.$$

Suppose z_1 and z_2 are integers satisfying equation $z_1^p - Dz_2^p = 1$ with $|z_2| > 1$. Then either $(z_1, z_2, D, p) = (18, 7, 17, 3)$ or $p > 163 \cdot 10^{12}$.

The bound for p is smaller than 10^6 in each case, so the previous result reduces our problem to only needing to consider finitely many small primes in each case. For $p = 3, 5$, or 7 on can solve the relevant Thue equations by Magma.

For $p \geq 11$ and for each value of m we are then left with at most one triple (A, B, C) and at most one value of p that we are unable to eliminate. Table 1 records these remaining values of p and corresponding triples. We note that one of A and B is equal to 1, and the other is exactly divisible by 2 in each case.

m	p	A	B	C
15	11	1	$2 \cdot 3 \cdot 23^{10}$	23
27	23	1	$2 \cdot 3 \cdot 47^{22}$	47
28	11	$2 \cdot 3 \cdot 23^{10}$	1	23
30	13	1	$2 \cdot 3 \cdot 53^{12}$	53
33	29	1	$2 \cdot 3 \cdot 59^{28}$	59
37	11	1	$2 \cdot 3 \cdot 67^{10}$	67
38	11	1	$2 \cdot 3 \cdot 23^{10}$	23
43	13	1	$2 \cdot 3 \cdot 79^{12}$	79
45	41	1	$2 \cdot 3 \cdot 83^{40}$	83
48	11	1	$2 \cdot 3 \cdot 89^{10}$	89

Table 1. Remaining cases after applying Theorem by Bartolomé and Mihăilescu and solving Thue equations for $p \leq 7$.

We rewrite our equation as

$$-1 - Dz_2^p + z_1^p = 0,$$

here $p \geq 11$ and $\mathrm{ord}_2(D) = 1$. The Frey curve we associate to this equation is

$$E: \; Y^2 = X(X+1)(X - Dz_2^p).$$

The conductor, N of E is then given by

$$N = \begin{cases} 2 \cdot \mathrm{Rad}_2(Dz_1z_2) & \text{if} \quad 2 \mid z_2, \\ 2^5 \cdot \mathrm{Rad}_2(Dz_1z_2) & \text{if} \quad 2 \nmid z_2. \end{cases}$$

Here, $\mathrm{Rad}_2(Dz_1z_2)$ denotes the product of all *odd* primes dividing Dz_1z_2.

We write $\overline{\rho}_{E,p}$ for the mod p Galois representation of E. Applying standard level-lowering results, we obtain that

$$\overline{\rho}_{E,p} \sim \overline{\rho}_{f,\mathfrak{p}},$$

for f a newform at level N_p, where

$$N_p = \begin{cases} 2 \cdot \mathrm{Rad}_2(D) & \text{if} \quad 2 \mid z_2, \\ 2^5 \cdot \mathrm{Rad}_2(D) & \text{if} \quad 2 \nmid z_2, \end{cases}$$

and $\mathfrak{p}$ a prime above p in the coefficient field of f.

Final steps

It remains to deal with the cases appearing in Table. Suppose we are in one of these cases, and let (y_1, y_2, y_3) be a non-zero solution to the original system of Thue equations. By rewriting ${y_i}^p$ as $-(-y_i)^p$ if necessary, we obtain an equation of the form

$$-1 - Dz_2^p + z_1^p = 0.$$

As described above, we attach a Frey curve E to this equation, and level lower so that $\overline{\rho}_{E,p} \sim \overline{\rho}_{f,\mathfrak{p}}$, for f a newform at level $2 \cdot \mathrm{Rad}_2(D)$ or $2^5 \cdot \mathrm{Rad}_2(D)$.

Now, if $\ell \mid y_1 y_2$ is a prime, then it must be a prime of multiplicative reduction for E, and by comparing traces of Frobenius, we have

$$\ell + 1 \equiv \pm c_\ell(f) \pmod{\mathfrak{p}},$$

where $c_\ell(f)$ denotes the ℓth Fourier coefficient of the newform f. It follows that

$$p \mid \mathrm{Norm}((\ell+1)^2 - c_\ell(f)^2)$$

We now search for a prime $\ell \equiv 1 \pmod{p}$, for which the system of Thue equations has a unique solution mod ℓ, *and* for which the divisibility relation does not hold. If the system has a unique solution mod ℓ, then this solution must be the reduction mod ℓ of the known global solution, for which $y_1 y_2 = 0$, so either $y_1 \equiv 0 \pmod{\ell}$ or $y_2 \equiv 0 \pmod{\ell}$. So $\ell \mid y_1 y_2$, and we have therefore obtained a contradiction if the previous divisibility relation does not hold.

m	p	f
15	11	138.2.a.d
27	23	282.2.a.e
28	11	138.2.a.d
30	13	318.2.a.g
33	29	354.2.a.h
37	11	402.2.a.g
38	11	—
43	13	474.2.a.e
45	41	498.2.a.g
48	11	534.2.a.f

Table 2. Remaining newforms. We use the notation of The LMFDB collaboration, The L-functions and modular forms database

For each newform f in each case we were able to find such a prime ℓ, apart from the cases listed in Table 2. For the remaining newforms in this Table, we find that for any prime $q \nmid 2D$ that we test,

$$p \mid \mathrm{Norm}(q + 1 - c_q(f)).$$

This suggests that the representation $\overline{\rho}_{f,\mathfrak{p}}$ is reducible, which would be a contradiction. We proceed by applying Proposition 2.2 from a paper by Bugeaud, Mignotte and Siksek (A multi-Frey approach to some multi-parameter families of Diophantine equations, Canadian Journal of Mathematics, 2008) to the newform f. We obtain that $p \mid \#E(\mathbb{F}_q)$ for any prime $q \nmid D$, and so E must have a rational subgroup of order p, a contradiction since $p \geq 11$.

Prologue–Two remarks

$$S_m^k(n) = \frac{n(n+1)\cdots(n+k-2)((m-2)n-m+k+2)}{k!}$$

$$S_m^2(n) = \frac{n\cdot((m-2)n-m+4)}{2} = \mathrm{Pol}_m(n)$$

$$\mathrm{Pol}_m(x) = y^z$$

in integers m, x, y, z with natural conditions.

Resolution for $m = 10$ A Unique Perfect Power Decagonal Number by Philippe Michaud-Rodgers, Bull. Austral Math. Soc. 2021.

Resolution for $m \in \{3, 5, 6, 8, 20\}$, Kim, Park, P, Bull. Austral Math. Soc. 2013.

Thank you very much
for your attention!