Irreducibility and Galois groups of Laguerre Polynomials

Ankita Jindal

Postdoctoral Fellow
ISI Bangalore Centre, India

April 28, 2023

Generalized Laguerre Polynomial

Definition

Let α and n be integers with $n \geq 1$. The Generalized Laguerre Polynomial of degree n with parameter α is denoted by $L_n^{(\alpha)}(x)$. It is defined by

$$L_n^{(\alpha)}(x) = \sum_{j=0}^{n} (-1)^{n-j} \frac{(n+\alpha)(n-1+\alpha)\cdots(j+1+\alpha)}{(n-j)!j!} x^j.$$

Generalized Laguerre Polynomial

Definition

Let α and n be integers with $n \geq 1$. The Generalized Laguerre Polynomial of degree n with parameter α is denoted by $L_n^{(\alpha)}(x)$. It is defined by

$$L_n^{(\alpha)}(x) = \sum_{j=0}^{n} (-1)^{n-j} \frac{(n+\alpha)(n-1+\alpha)\cdots(j+1+\alpha)}{(n-j)!\,j!} x^j.$$

- The leading coefficient of $L_n^{(\alpha)}(x)$ is $\frac{1}{n!}$.

Generalized Laguerre Polynomial

Definition

Let α and n be integers with $n \geq 1$. The Generalized Laguerre Polynomial of degree n with parameter α is denoted by $L_n^{(\alpha)}(x)$. It is defined by

$$L_n^{(\alpha)}(x) = \sum_{j=0}^{n} (-1)^{n-j} \frac{(n+\alpha)(n-1+\alpha)\cdots(j+1+\alpha)}{(n-j)!j!} x^j.$$

- The leading coefficient of $L_n^{(\alpha)}(x)$ is $\frac{1}{n!}$.

In what follows, a polynomial $f(x) \in \mathbb{Q}[x]$ will be called irreducible, if it is irreducible over $\mathbb{Q}$.

Special types of Laguerre polynomials

- For $\alpha = 0$, $L_n^{(0)}(x)$ has a simpler form given by

$$L_n^{(0)}(x) = \sum_{j=0}^{n} (-1)^{n-j} \frac{n(n-1)\cdots(j+1)}{(n-j)!j!} x^j = \sum_{j=0}^{n} \frac{(-1)^{n-j}}{j!} \binom{n}{j} x^j.$$

Special types of Laguerre polynomials

- For $\alpha = 0$, $L_n^{(0)}(x)$ has a simpler form given by

$$L_n^{(0)}(x) = \sum_{j=0}^{n} (-1)^{n-j} \frac{n(n-1)\cdots(j+1)}{(n-j)!j!} x^j = \sum_{j=0}^{n} \frac{(-1)^{n-j}}{j!} \binom{n}{j} x^j.$$

$L_n^{(0)}(x)$ is called the classical Laguerre polynomial of degree n.

Special types of Laguerre polynomials

- For $\alpha = 0$, $L_n^{(0)}(x)$ has a simpler form given by

$$L_n^{(0)}(x) = \sum_{j=0}^{n} (-1)^{n-j} \frac{n(n-1)\cdots(j+1)}{(n-j)!j!} x^j = \sum_{j=0}^{n} \frac{(-1)^{n-j}}{j!} \binom{n}{j} x^j.$$

$L_n^{(0)}(x)$ is called the classical Laguerre polynomial of degree n.

- For $\alpha = -n-1$, $L_n^{(-n-1)}(x)$ also has a simple form given by

$$\begin{aligned} L_n^{(-n-1)}(x) &= \sum_{j=0}^{n} (-1)^{n-j} \frac{(-1)(-2)\cdots(-n+j)}{(n-j)!j!} x^j \\ &= \sum_{j=0}^{n} \frac{x^j}{j!}. \end{aligned}$$

So $L_n^{(-n-1)}(x)$ is the n^{th} Taylor polynomial of the exponential function.

Well known results of I. Schur

Theorem (I. Schur, 1930)

$L_n^{(-n-1)}(x)$ is irreducible for each $n \geq 1$. The Galois group of $L_n^{(-n-1)}(x)$ is the alternating group A_n of degree n if $n \equiv 0 \pmod 4$, and it is the symmetric group S_n of degree n if $n \not\equiv 0 \pmod 4$.

Well known results of I. Schur

Theorem (I. Schur, 1930)

$L_n^{(-n-1)}(x)$ is irreducible for each $n \geq 1$. The Galois group of $L_n^{(-n-1)}(x)$ is the alternating group A_n of degree n if $n \equiv 0 \pmod 4$, and it is the symmetric group S_n of degree n if $n \not\equiv 0 \pmod 4$.

Theorem (I. Schur, 1930)

$L_n^{(0)}(x)$ is irreducible for each $n \geq 1$. The Galois group of $L_n^{(0)}(x)$ is S_n for each n.

Theorem (I. Schur, 1930)

$L_n^{(1)}(x)$ is irreducible for each $n \geq 1$. The Galois group of $L_n^{(1)}(x)$ is A_n if n is odd or $n+1$ is an odd square and S_n otherwise.

Bessel polynomials

- The Bessel polynomial of degree n is defined by

$$y_n(x) = \sum_{j=0}^{n} \frac{(n+j)!}{(n-j)!j!} \left(\frac{x}{2}\right)^j.$$

Bessel polynomials

- The Bessel polynomial of degree n is defined by

$$y_n(x) = \sum_{j=0}^{n} \frac{(n+j)!}{(n-j)!\,j!} \left(\frac{x}{2}\right)^j.$$

- It can be easily checked that

$$x^n y_n\left(\frac{2}{x}\right) = n!L_n^{(-2n-1)}(x) = \sum_{j=0}^{n} \frac{(2n-j)!}{(n-j)!\,j!} x^j.$$

Bessel polynomials

- The Bessel polynomial of degree n is defined by

$$y_n(x) = \sum_{j=0}^{n} \frac{(n+j)!}{(n-j)!j!} \left(\frac{x}{2}\right)^j.$$

- It can be easily checked that

$$x^n y_n\left(\frac{2}{x}\right) = n!L_n^{(-2n-1)}(x) = \sum_{j=0}^{n} \frac{(2n-j)!}{(n-j)!j!} x^j.$$

- The irreducibility for n^{th} degree Bessel polynomial for each $n \geq 1$ was proved by M. Filaseta and O. Trifonov in 2002.

Irreducibility of $L_n^{(\alpha)}(x)$ over $\mathbb{Q}$

- Let $n \geq 2$ be an integer. For $\alpha = -a$ where $1 \leq a < n$, it can be easily seen that

$$\begin{aligned} L_n^{(-a)}(x) &= \sum_{j=0}^{n} (-1)^{n-j} \frac{(n-a)(n-1-a)\cdots(j+1-a)}{(n-j)!j!} x^j \\ &= x^a L_{n-a}^{(a)}(x). \end{aligned}$$

Hence $L_n^{(\alpha)}(x)$ is reducible for $\alpha = -a$ where $1 \leq a < n$.

Irreducibility of $L_n^{(\alpha)}(x)$ over $\mathbb{Q}$

- Let $n \geq 2$ be an integer. For $\alpha = -a$ where $1 \leq a < n$, it can be easily seen that

$$\begin{aligned} L_n^{(-a)}(x) &= \sum_{j=0}^{n} (-1)^{n-j} \frac{(n-a)(n-1-a)\cdots(j+1-a)}{(n-j)!j!} x^j \\ &= x^a L_{n-a}^{(a)}(x). \end{aligned}$$

Hence $L_n^{(\alpha)}(x)$ is reducible for $\alpha = -a$ where $1 \leq a < n$.

- One can also check that

$$\begin{aligned} L_2^{(2)}(x) &= \frac{1}{2}(x-2)(x-6), \\ L_2^{(23)}(x) &= \frac{1}{2}(x-20)(x-30), \\ L_4^{(23)}(x) &= \frac{1}{24}(x-30)(x^3-78x^2+1872x-14040). \end{aligned}$$

Irreducibility of $L_n^{(\alpha)}(x)$ over $\mathbb{Q}$

It is known that $L_n^{(\alpha)}(x)$ is irreducible for the following values of α.

I. Schur (1930), R. F. Coleman (1987) : $\alpha = -n-1$

Irreducibility of $L_n^{(\alpha)}(x)$ over $\mathbb{Q}$

It is known that $L_n^{(\alpha)}(x)$ is irreducible for the following values of α.

I. Schur (1930), R. F. Coleman (1987) : $\alpha = -n-1$

F. Hajir (1995)

Irreducibility of $L_n^{(\alpha)}(x)$ over $\mathbb{Q}$

It is known that $L_n^{(\alpha)}(x)$ is irreducible for the following values of α.

I. Schur (1930), R. F. Coleman (1987)	: $\alpha = -n-1$
F. Hajir (1995)	: $\alpha = -n-2$

Irreducibility of $L_n^{(\alpha)}(x)$ over $\mathbb{Q}$

It is known that $L_n^{(\alpha)}(x)$ is irreducible for the following values of α.

I. Schur (1930), R. F. Coleman (1987) : $\alpha = -n-1$

F. Hajir (1995) : $\alpha = -n-2$

M. Filaseta, O. Trifonov (2002)

Irreducibility of $L_n^{(\alpha)}(x)$ over $\mathbb{Q}$

It is known that $L_n^{(\alpha)}(x)$ is irreducible for the following values of α.

I. Schur (1930), R. F. Coleman (1987)	:	$\alpha = -n-1$
F. Hajir (1995)	:	$\alpha = -n-2$
M. Filaseta, O. Trifonov (2002)	:	$\alpha = -2n-1$

Irreducibility of $L_n^{(\alpha)}(x)$ over $\mathbb{Q}$

It is known that $L_n^{(\alpha)}(x)$ is irreducible for the following values of α.

I. Schur (1930), R. F. Coleman (1987)	: $\alpha = -n-1$
F. Hajir (1995)	: $\alpha = -n-2$
M. Filaseta, O. Trifonov (2002)	: $\alpha = -2n-1$
E. A. Sell (2004)	

Irreducibility of $L_n^{(\alpha)}(x)$ over $\mathbb{Q}$

It is known that $L_n^{(\alpha)}(x)$ is irreducible for the following values of α.

I. Schur (1930), R. F. Coleman (1987)	: $\alpha = -n-1$
F. Hajir (1995)	: $\alpha = -n-2$
M. Filaseta, O. Trifonov (2002)	: $\alpha = -2n-1$
E. A. Sell (2004)	: $\alpha = -n-3$

Irreducibility of $L_n^{(\alpha)}(x)$ over $\mathbb{Q}$

It is known that $L_n^{(\alpha)}(x)$ is irreducible for the following values of α.

I. Schur (1930), R. F. Coleman (1987) : $\alpha = -n-1$

F. Hajir (1995) : $\alpha = -n-2$

M. Filaseta, O. Trifonov (2002) : $\alpha = -2n-1$

E. A. Sell (2004) : $\alpha = -n-3$

F. Hajir (2009)

Irreducibility of $L_n^{(\alpha)}(x)$ over $\mathbb{Q}$

It is known that $L_n^{(\alpha)}(x)$ is irreducible for the following values of α.

I. Schur (1930), R. F. Coleman (1987)	: $\alpha = -n-1$
F. Hajir (1995)	: $\alpha = -n-2$
M. Filaseta, O. Trifonov (2002)	: $\alpha = -2n-1$
E. A. Sell (2004)	: $\alpha = -n-3$
F. Hajir (2009)	: $\alpha = -n-1-r$ for $r \in [3,8]$

Irreducibility of $L_n^{(\alpha)}(x)$ over $\mathbb{Q}$

It is known that $L_n^{(\alpha)}(x)$ is irreducible for the following values of α.

I. Schur (1930), R. F. Coleman (1987)	: $\alpha = -n-1$
F. Hajir (1995)	: $\alpha = -n-2$
M. Filaseta, O. Trifonov (2002)	: $\alpha = -2n-1$
E. A. Sell (2004)	: $\alpha = -n-3$
F. Hajir (2009)	: $\alpha = -n-1-r$ for $r \in [3,8]$
M. Filaseta, T. Kidd, O. Trifonov (2012)	

Irreducibility of $L_n^{(\alpha)}(x)$ over $\mathbb{Q}$

It is known that $L_n^{(\alpha)}(x)$ is irreducible for the following values of α.

I. Schur (1930), R. F. Coleman (1987)	: $\alpha = -n-1$
F. Hajir (1995)	: $\alpha = -n-2$
M. Filaseta, O. Trifonov (2002)	: $\alpha = -2n-1$
E. A. Sell (2004)	: $\alpha = -n-3$
F. Hajir (2009)	: $\alpha = -n-1-r$ for $r \in [3,8]$
M. Filaseta, T. Kidd, O. Trifonov (2012)	: $\alpha = n$ with $n \equiv 2 \pmod 4$

Irreducibility of $L_n^{(\alpha)}(x)$ over $\mathbb{Q}$

It is known that $L_n^{(\alpha)}(x)$ is irreducible for the following values of α.

I. Schur (1930), R. F. Coleman (1987)	: $\alpha = -n-1$
F. Hajir (1995)	: $\alpha = -n-2$
M. Filaseta, O. Trifonov (2002)	: $\alpha = -2n-1$
E. A. Sell (2004)	: $\alpha = -n-3$
F. Hajir (2009)	: $\alpha = -n-1-r$ for $r \in [3,8]$
M. Filaseta, T. Kidd, O. Trifonov (2012)	: $\alpha = n$ with $n \equiv 2 \pmod 4$
S. Nair, T. N. Shorey (2015)	

Irreducibility of $L_n^{(\alpha)}(x)$ over $\mathbb{Q}$

It is known that $L_n^{(\alpha)}(x)$ is irreducible for the following values of α.

I. Schur (1930), R. F. Coleman (1987)	: $\alpha = -n-1$
F. Hajir (1995)	: $\alpha = -n-2$
M. Filaseta, O. Trifonov (2002)	: $\alpha = -2n-1$
E. A. Sell (2004)	: $\alpha = -n-3$
F. Hajir (2009)	: $\alpha = -n-1-r$ for $r \in [3,8]$
M. Filaseta, T. Kidd, O. Trifonov (2012)	: $\alpha = n$ with $n \equiv 2 \pmod 4$
S. Nair, T. N. Shorey (2015)	: $\alpha = -n-1-r$ for $r \in [9,22]$

Irreducibility of $L_n^{(\alpha)}(x)$ over $\mathbb{Q}$

It is known that $L_n^{(\alpha)}(x)$ is irreducible for the following values of α.

I. Schur (1930), R. F. Coleman (1987)	: $\alpha = -n-1$
F. Hajir (1995)	: $\alpha = -n-2$
M. Filaseta, O. Trifonov (2002)	: $\alpha = -2n-1$
E. A. Sell (2004)	: $\alpha = -n-3$
F. Hajir (2009)	: $\alpha = -n-1-r$ for $r \in [3,8]$
M. Filaseta, T. Kidd, O. Trifonov (2012)	: $\alpha = n$ with $n \equiv 2 \pmod 4$
S. Nair, T. N. Shorey (2015)	: $\alpha = -n-1-r$ for $r \in [9,22]$

- $\alpha = -n-1-r$ for $r \in [23,92]$
- $\alpha = -2n-\beta$ for $\beta \in [0,4]$
- $\alpha = n+\gamma$ for $\gamma \in [-6,3]$

$\alpha = -n-1-r$ for $23 \leq r \leq 92$

Theorem 1. (A. Jindal, S. Laishram, R. Sarma), 2018

For integers $n \geq 1$ and $23 \leq r \leq 60$, $L_n^{(-n-1-r)}(x)$ is irreducible.

$\alpha = -n-1-r$ for $23 \leq r \leq 92$

Theorem 1. (A. Jindal, S. Laishram, R. Sarma), 2018

For integers $n \geq 1$ and $23 \leq r \leq 60$, $L_n^{(-n-1-r)}(x)$ is irreducible.

Theorem. (T. N. Shorey, S. B. Sinha), 2022

For integers $n \geq 3$ and $r \leq 92$, $L_n^{(-n-1-r)}(x)$ is either irreducible or $L_n^{(-n-1-r)}(x)$ is a product of a linear polynomial with an irreducible polynomial of degree $n-1$.

$\alpha = -n-1-r$ for $23 \leq r \leq 92$

Theorem 1. (A. Jindal, S. Laishram, R. Sarma), 2018

For integers $n \geq 1$ and $23 \leq r \leq 60$, $L_n^{(-n-1-r)}(x)$ is irreducible.

Theorem. (T. N. Shorey, S. B. Sinha), 2022

For integers $n \geq 3$ and $r \leq 92$, $L_n^{(-n-1-r)}(x)$ is either irreducible or $L_n^{(-n-1-r)}(x)$ is a product of a linear polynomial with an irreducible polynomial of degree $n-1$.

Theorem 2. (A. Jindal, S. Nair, T. N. Shorey), 2023

For integers $n \geq 1$ and $61 \leq r \leq 92$, $L_n^{(-n-1-r)}(x)$ is irreducible.

$\alpha = -2n - \beta$ for $0 \leq \beta \leq 4$

Theorem 3. (A. Jindal, S. Laishram), 2022

Let $n \geqslant 1$ be an integer.

(i) For an integer $\beta \in [1,4]$, $L_n^{(-2n-\beta)}(x)$ is irreducible.

(ii) $L_n^{(-2n)}(x)$ is either irreducible or $L_n^{(-2n)}(x)$ is a product of a linear polynomial with an irreducible polynomial of degree $n-1$. Further if

$$2^\ell || n \text{ with } \ell \leq n^{1/2},$$

then

$$L_n^{(-2n)}(x) \text{ is irreducible.}$$

In particular, $L_n^{(-2n)}(x)$ is irreducible for odd n.

(iii) $L_n^{(-2n)}(x)$ is irreducible if n is a power of 2.

Theorem 4. (A. Jindal, S.Laishram), 2023

Let $n \geqslant 1$ be an integer.

(i) For $\gamma \in \{-2,-1,2,3\}$, $L_n^{(n+\gamma)}(x)$ is irreducible.

(ii) For $\gamma = 1$, $L_n^{(n+1)}(x)$ is irreducible for $n \neq 4$. In fact,

$$L_4^{(5)}(x) = \frac{1}{5!}(x+6)(x^3+30x^2+252x+504).$$

(iii) For an integer $\gamma \in [-6,-3]$ and $n \notin \left[\frac{|\gamma|}{2}, |\gamma|-1\right]$, $L_n^{(n+\gamma)}(x)$ is irreducible.

(iv) $L_n^{(n)}(x)$ is either irreducible or is a product of a linear polynomial and an irreducible polynomial of degree $n-1$. Further, if

$$2^\ell || n \text{ with } \ell \le n^{1/2},$$

then $L_n^{(n)}(x)$ is irreducible. In particular, $L_n^{(n)}(x)$ is irreducible for odd n. Also $L_n^{(n)}(x)$ is irreducible, if n is a power of 2.

p-Newton polygon

For a prime p and a non-zero integer a, $v_p(a)$ will stand for the highest power of p dividing a. We set $v_p(0) = \infty$.

Let p be a prime number.

- Let $f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 \in \mathbb{Z}[x]$ with $a_0 a_n \neq 0$.
- Let P_i stand for the point in the plane having the coordinates $(i, v_p(a_{n-i}))$ when $a_i \neq 0$, $0 \leq i \leq n$. We consider the set

$$S = \{(i, v_p(a_{n-i})) \mid 0 \leq i \leq n,\ a_{n-i} \neq 0\}.$$

- Let μ_{ij} denote the slope of the line joining P_i and P_j if $a_i a_j \neq 0$.

p-Newton polygon

For a prime p and a non-zero integer a, $v_p(a)$ will stand for the highest power of p dividing a. We set $v_p(0) = \infty$.

Let p be a prime number.

- Let $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 \in \mathbb{Z}[x]$ with $a_0 a_n \neq 0$.
- Let P_i stand for the point in the plane having the coordinates $(i, v_p(a_{n-i}))$ when $a_i \neq 0$, $0 \leq i \leq n$. We consider the set

$$S = \{(i, v_p(a_{n-i})) \mid 0 \leq i \leq n,\ a_{n-i} \neq 0\}.$$

- Let μ_{ij} denote the slope of the line joining P_i and P_j if $a_i a_j \neq 0$.

- Let i_1 be the largest index $0 < i_1 \leq n$ such that

$$\mu_{0i_1} = \min\{\mu_{0j} \mid 0 < j \leq n,\ a_{n-j} \neq 0\}.$$

If $i_1 < n$, let i_2 be the largest index $i_1 < i_2 \leq n$ such that

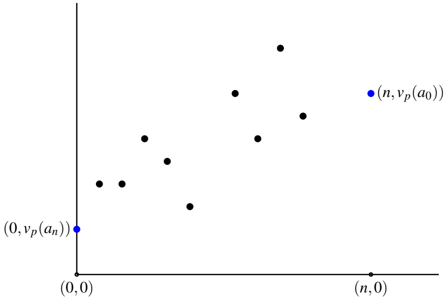
$$\mu_{i_1 i_2} = \min\{\mu_{i_1 j} \mid i_1 < j \leq n,\ a_{n-j} \neq 0\}.$$

and so on.

- The p-Newton polygon of $f(x)$ is the polygonal path having segments $P_0P_{i_1}, P_{i_1}P_{i_2}, \ldots, P_{i_{k-1}}P_{i_k}$ with $i_k = n$.
- These segments are called the edges of the p-Newton polygon of $f(x)$ and their slopes form a strictly increasing sequence.

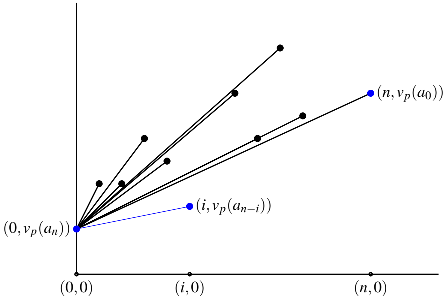
If each $a_i \neq 0$,

$$\begin{array}{ccccccccc} f(x) = a_n x^n + & a_{n-1}x^{n-1} + & a_{n-2}x^{n-2} + & \cdots & + a_i x^i + & \cdots & + a_0 \\ \downarrow & \downarrow & \downarrow & & \downarrow & & \downarrow \\ S = \{(0, v_p(a_n)), & (1, v_p(a_{n-1})), & (2, v_p(a_{n-2})), & \ldots & ,(i, v_p(a_{n-i})), & \ldots, & (n, v_p(a_0))\} \\ \shortparallel & \shortparallel & \shortparallel & & \shortparallel & & \shortparallel \\ P_0 & P_1 & P_2 & & P_i & & P_n \end{array}$$



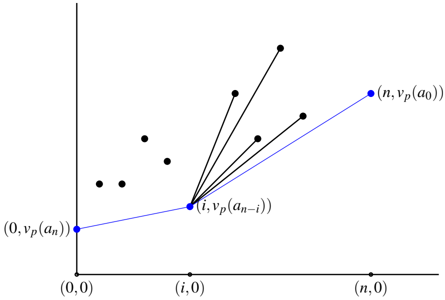
If each $a_i \neq 0$,

$$\begin{array}{ccccccc} f(x) = a_n x^n + & a_{n-1}x^{n-1} + & a_{n-2}x^{n-2} + & \cdots & + a_i x^i + & \cdots & + a_0 \\ \downarrow & \downarrow & \downarrow & & \downarrow & & \downarrow \\ S = \{(0, v_p(a_n)), & (1, v_p(a_{n-1})), & (2, v_p(a_{n-2})), & \ldots & ,(i, v_p(a_{n-i})), & \ldots, & (n, v_p(a_0))\} \\ \shortparallel & \shortparallel & \shortparallel & & \shortparallel & & \shortparallel \\ P_0 & P_1 & P_2 & & P_i & & P_n \end{array}$$



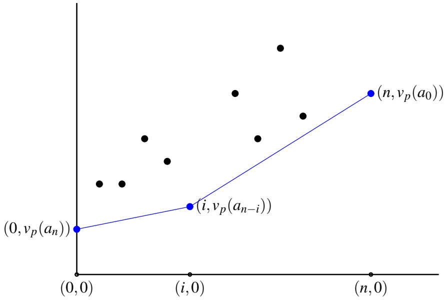
If each $a_i \neq 0$,

$$
\begin{array}{ccccccc}
f(x) = a_n x^n + & a_{n-1}x^{n-1} + & a_{n-2}x^{n-2} + & \cdots & + a_i x^i + & \cdots & + a_0 \\
\downarrow & \downarrow & \downarrow & & \downarrow & & \downarrow \\
S = \{(0, v_p(a_n)), & (1, v_p(a_{n-1})), & (2, v_p(a_{n-2})), & \ldots & ,(i, v_p(a_{n-i})), & \ldots, & (n, v_p(a_0))\} \\
\shortparallel & \shortparallel & \shortparallel & & \shortparallel & & \shortparallel \\
P_0 & P_1 & P_2 & & P_i & & P_n
\end{array}
$$



If each $a_i \neq 0$,

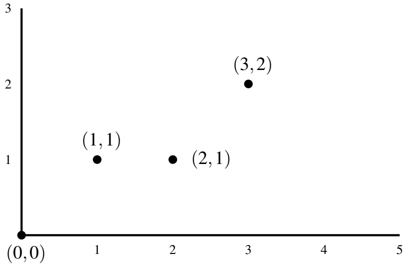
$$
\begin{array}{ccccccc}
f(x) = a_n x^n + & a_{n-1}x^{n-1} + & a_{n-2}x^{n-2} + & \cdots & + a_i x^i + & \cdots & + a_0 \\
\downarrow & \downarrow & \downarrow & & \downarrow & & \downarrow \\
S = \{(0, v_p(a_n)), & (1, v_p(a_{n-1})), & (2, v_p(a_{n-2})), & \ldots & ,(i, v_p(a_{n-i})), & \ldots, & (n, v_p(a_0))\} \\
\shortparallel & \shortparallel & \shortparallel & & \shortparallel & & \shortparallel \\
P_0 & P_1 & P_2 & & P_i & & P_n
\end{array}
$$



Example

Let $p = 3$. Consider the polynomial $f(x) = x^3 + 3x^2 + 12x + 9$.

$$f(x) = x^3 \ + \ 3x^2 \ + 12x \ + \ 9$$
$$\downarrow \quad \downarrow \quad \downarrow \quad \downarrow$$
$$S = \{(0,0),\ (1,1),\ (2,1),\ (3,2)\}$$

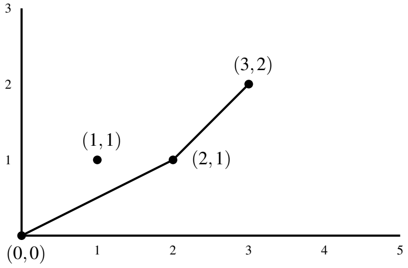


3-Newton Polygon of $f(x) = x^3 + 3x^2 + 12x + 9$

Example

Let $p = 3$. Consider the polynomial $f(x) = x^3 + 3x^2 + 12x + 9$.

$$f(x) = x^3 \ + \ 3x^2 \ + 12x \ + \ 9$$
$$\downarrow \quad \downarrow \quad \downarrow \quad \downarrow$$
$$S = \{(0,0),\ (1,1),\ (2,1),\ (3,2)\}$$



3-Newton Polygon of $f(x) = x^3 + 3x^2 + 12x + 9$

p-Newton polygon of Eisenstein polynomial

Definition.

Let p be a prime. Let

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0 \in \mathbb{Z}[x]$$

such that $p \nmid a_n,\ p | a_i$ for $0 \leq i \leq n-1,\ p^2 \nmid a_0$. Such a polynomial is said to be an Eisenstein polynomial with respect to p.

p-Newton polygon of Eisenstein polynomial

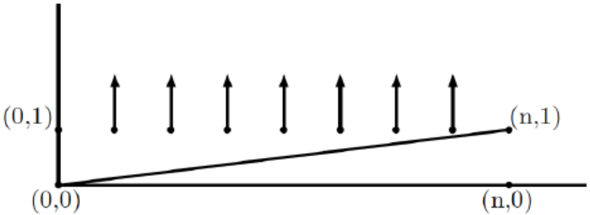
Definition.

Let p be a prime. Let

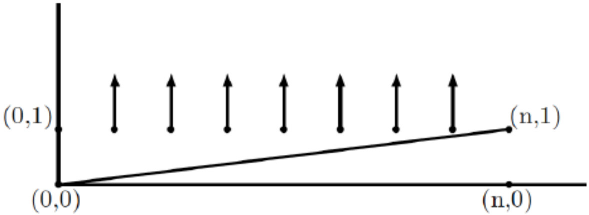
$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0 \in \mathbb{Z}[x]$$

such that $p \nmid a_n,\ p|a_i$ for $0 \leq i \leq n-1,\ p^2 \nmid a_0$. Such a polynomial is said to be an Eisenstein polynomial with respect to p.

- Then p-Newton polygon of $f(x)$ consists of only one edge which has slope $\frac{1}{n}$.



p-Newton polygon of Eisenstein polynomial

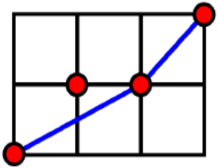


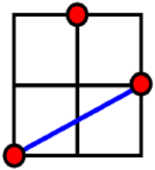
Restatement of Eisenstein Irreducibility Criterion:

- Let $f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0 \in \mathbb{Z}[x]$.
- Assume that the p-Newton polygon of $f(x)$ for some prime p has only one edge with vertices $\{(0,0),(n,1)\}$.
- Then $f(x)$ is irreducible over $\mathbb{Q}$.

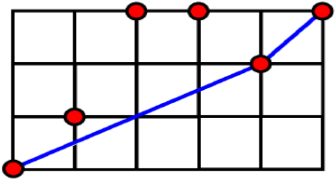
Example

Let $p = 3$.





3-Newton polygons of $f(x) = x^3 + 3x^2 + 12x + 9$ and $g(x) = 2x^2 + 9x + 3$



3-Newton polygon of $f(x)g(x) = 2x^5 + 15x^4 + 54x^3 + 135x^2 + 117x + 27$

Dumas' result on the p-Newton polygon of product of polynomials

Theorem. (G. Dumas), 1906

Let $g(x), h(x) \in \mathbb{Z}[x]$ with $g(0)h(0) \neq 0$, and let p be a prime. Let $p^t \geq 1$ be the highest power of p dividing the leading coefficient of $g(x)h(x)$. Then the p-Newton polygon of $g(x)h(x)$ can be formed by constructing a polygonal path beginning at $(0,t)$ and using translates of the edges in the p-Newton polygons of $g(x)$ and $h(x)$ in the increasing order of slopes.

Dumas' result on the p-Newton polygon of product of polynomials

Theorem. (G. Dumas), 1906

Let $g(x), h(x) \in \mathbb{Z}[x]$ with $g(0)h(0) \neq 0$, and let p be a prime. Let $p^t \geq 1$ be the highest power of p dividing the leading coefficient of $g(x)h(x)$. Then the p-Newton polygon of $g(x)h(x)$ can be formed by constructing a polygonal path beginning at $(0,t)$ and using translates of the edges in the p-Newton polygons of $g(x)$ and $h(x)$ in the increasing order of slopes.

Note that Eisenstein Irreducibility Criterion follows immediately from the above theorem. Because for polynomial $f(x)$ of degree n which is Eisenstein with respect to p, the p-Newton polygon of $f(x)$ consists of a single edge without any point with integer entries other than $(0,0)$ and $(n,1)$.

Filaseta's Criterion

• If a polynomial $f(x)$ of degree n is reducible, then $f(x)$ necessarily has a factor of degree $k \in [1, \frac{n}{2}]$.

Filaseta's Criterion

- If a polynomial $f(x)$ of degree n is reducible, then $f(x)$ necessarily has a factor of degree $k \in [1, \frac{n}{2}]$.

Theorem (M. Filaseta), 1995

Let n be a positive integer and p be a prime. Let k and ℓ be integers with $0 \leq \ell < k \leq \frac{n}{2}$. Let $f(x) = a_n x^n + \cdots + a_1 x + a_0 \in \mathbb{Z}[x]$ be a polynomial of degree n with non-zeo constant term. Suppose that

(i) $p \nmid a_n$,

(ii) $p \mid a_j$ for all $j \in \{0, 1, \ldots, n - \ell - 1\}$, and

(iii) the right-most edge of the p-Newton polygon of $f(x)$ has slope $< \frac{1}{k}$.

Then $f(x)$ does not have a factor over $\mathbb{Q}$ with degree in the interval $[\ell + 1, k]$.

Note that Eisenstein's irreducibility criterion follows from Filaseta's result by taking $\ell = 0$ and $k = \frac{n}{2}$.

Another application of p-Newton polygons

Theorem 5. (A. Jindal, S. Laishram), 2023

Let n be a positive integer and let p be a prime. Let k, ℓ and $\ell^{'}$ be integers with $0 \leq \ell^{'} \leq \ell < k \leq \frac{n}{2}$. Let $f(x) = a_n x^n + \cdots + a_1 x + a_0 \in \mathbb{Z}[x]$ be a polynomial of degree n with non-zero constant term. Suppose that

(i) $p \nmid a_n$

(ii) $p | a_j$ for all $j \in \{0, 1, \ldots, n - (\ell - \ell^{'}) - 1\}$,

(iii) $v_p(a_{\ell+1}) \geqslant v_p(a_0)$ and

$$v_p(a_\ell) = v_p(a_{\ell-1}) = \cdots = v_p(a_{\ell'}) < v_p(a_{\ell'-1}) = v_p(a_{\ell'-2}) = \cdots = v_p(a_1) = v_p(a_0),$$

(iv) $\max\limits_{\ell+1 < j \leqslant n} \frac{v_p(a_0) - v_p(a_j)}{j} < \frac{1}{k}$.

Then the polynomial $f(x)$ does not have a factor over $\mathbb{Q}$ with degree in the interval $[\ell + 1, k]$.

Definition

For an integer $m > 1$, let $P(m)$ denotes the greatest prime factor of m. We take $P(1) = 1$.

Theorem. (S. Nair, T. N. Shorey), 2016

Let $k \geq 2$ and $n \geq 5k$ be integers. Then

$$P(n(n-1)\cdots(n-k+1)) > 4.42k$$

except for

$$\begin{aligned}(n,k) \in \{&(10,2),(15,2),(16,2),(21,2),(25,2),(28,2),(36,2),(49,2),(50,2),\\ &(64,2),(81,2),(126,2),(225,2),(2401,2),(4375,2),(15,3),(16,3),\\ &(22,3),(26,3),(27,3),(28,3),(50,3),(56,3),(65,3),(66,3),(100,3),\\ &(352,3),(27,4),(28,4),(35,4),(36,4),(51,4),(52,4),(66,4),(28,5),\\ &(36,5),(52,5),(35,7),(36,7)\}.\end{aligned}$$

Some observations from Lehmer's Table

In 1964, D. H. Lehmer gave the tables, namely Table IA, Table IIA and Table IIIA, along with many other tables where

- Table IA consists of all the integers $x > 1$ such that $P(x(x-1)) \leqslant 41$.
- Table IIA consists of all the odd integers $x > 1$ such that $P(x(x-2)) \leqslant 31$.
- Table IIIA consists of all the odd integers $x > 3$ such that $P(x(x-4)) \leqslant 31$.

Using these table, we have

- $$P(x(x-1)) \geqslant \begin{cases} 7 & \text{if } x > 81, \\ 11 & \text{if } x > 64 \text{ and } x \notin \{81, 126, 225, 2401, 4375\}, \\ 43 & \text{if } x > 6.4 \times 10^{10}. \end{cases}$$

Some observations from Lehmer's Table

In 1964, D. H. Lehmer gave the tables, namely Table IA, Table IIA and Table IIIA, along with many other tables where

- Table IA consists of all the integers $x > 1$ such that $P(x(x-1)) \leqslant 41$.
- Table IIA consists of all the odd integers $x > 1$ such that $P(x(x-2)) \leqslant 31$.
- Table IIIA consists of all the odd integers $x > 3$ such that $P(x(x-4)) \leqslant 31$.

Using these table, we have

- $$P(x(x-1)) \geqslant \begin{cases} 7 & \text{if } x > 81, \\ 11 & \text{if } x > 64 \text{ and } x \notin \{81, 126, 225, 2401, 4375\}, \\ 43 & \text{if } x > 6.4 \times 10^{10}. \end{cases}$$

- $P(x(x-2)) \geqslant 37$ if $x > 287080367$ and x is odd.

Some observations from Lehmer's Table

In 1964, D. H. Lehmer gave the tables, namely Table IA, Table IIA and Table IIIA, along with many other tables where

- Table IA consists of all the integers $x > 1$ such that $P(x(x-1)) \leqslant 41$.
- Table IIA consists of all the odd integers $x > 1$ such that $P(x(x-2)) \leqslant 31$.
- Table IIIA consists of all the odd integers $x > 3$ such that $P(x(x-4)) \leqslant 31$.

Using these table, we have

- $$P(x(x-1)) \geqslant \begin{cases} 7 & \text{if } x > 81, \\ 11 & \text{if } x > 64 \text{ and } x \notin \{81, 126, 225, 2401, 4375\}, \\ 43 & \text{if } x > 6.4 \times 10^{10}. \end{cases}$$

- $P(x(x-2)) \geqslant 37$ if $x > 287080367$ and x is odd.

- $P(x(x-4)) \geqslant 37$ if $x > 10439037$ and x is odd.

Luca and Najman's Table

Theorem. (F. Luca, F. Najman) 2011

For $2 \leqslant k \leqslant 9$ and $n > n_k$, we have $P(n(n-1)\cdots(n-k+1)) \geqslant 101$ where n_k are given by

k	2	3	4	5	6	7	8	9
n_k	9591468737351909376	407498960	97527	7569	7569	4902	4902	292

- $n = 292$ is the largest positive integer n satisfying

$$P(n(n-1)\cdots(n-8)) < 101.$$

Inverse Galois Problem

The following problem was posed in the early 19^{th} century:

Given a finite group G, whether there exists a Galois extension of $\mathbb{Q}$ whose Galois group is G? This is called the Inverse Problem of Galois Theory and is one of the most challenging problems in mathematics. It is still open in general.

Inverse Galois Problem

The following problem was posed in the early 19^{th} century:

Given a finite group G, whether there exists a Galois extension of $\mathbb{Q}$ whose Galois group is G? This is called the Inverse Problem of Galois Theory and is one of the most challenging problems in mathematics. It is still open in general.

The first systematic approach to a solution of the inverse Galois problem goes back to Hilbert (1892) who proved that there exists an irreducible polynomial of degree n over $\mathbb{Q}$ whose Galois group is S_n.

Inverse Galois Problem

The following problem was posed in the early 19^{th} century:

Given a finite group G, whether there exists a Galois extension of $\mathbb{Q}$ whose Galois group is G? This is called the Inverse Problem of Galois Theory and is one of the most challenging problems in mathematics. It is still open in general.

The first systematic approach to a solution of the inverse Galois problem goes back to Hilbert (1892) who proved that there exists an irreducible polynomial of degree n over $\mathbb{Q}$ whose Galois group is S_n.

Since every finite group is isomorphic to a subgroup of S_n, in view of fundamental theorem of Galois Theory it follows that every finite group G is the Galois group of a Galois extension of algebraic number fields.

Galois group of $L_n^{(\alpha)}(x)$ over $\mathbb{Q}$

Theorem 6. (A. Jindal, S. Laishram, R. Sarma), 2018

For integers $n \geq 1$ and $23 \leq r \leq 60$, the Galois group of $L_n^{(-n-1-r)}(x)$ is S_n unless

$$(n,r) \in \{(4,24),(5,28),(24,25),(25,24),(28,23),(28,29),(32,33),(33,36),(36,37),(40,41),(44,45),(48,49),(48,51),(49,48),(49,50),(52,53),(56,57)\},$$

in which case its Galois group is A_n.

Theorem 7. (A. Jindal, S. Nair, T. N. Shorey), 2023

For integers $n \geq 1$ and $61 \leq r \leq 92$, the Galois group of $L_n^{(-n-1-r)}(x)$ is S_n unless

$$(n,r) \in \{(60,61),(61,64),(64,65),(68,69),(72,73),(76,77),(80,81),(81,80),(84,85),(88,89),(92,93),(96,97),(97,98),(97,100)\},$$

in which case its Galois group is A_n.

Galois group of $L_n^{(\alpha)}(x)$ over $\mathbb{Q}$

• The rational part of an element $\delta = a + b\sqrt{2} \in \mathbb{Q}(\sqrt{2})$ with $a, b \in \mathbb{Q}$ is defined to be a.

Theorem 8. (A. Jindal, S. Laishram), 2022

Let $n \geq 1$ be an integer and $\beta \in \{0, 2, 4\}$. The associated Galois group of $L_n^{(-2n-\beta)}(x)$ is A_n if and only if one of the following conditions is satisfied:

(i) $\beta = 0$ and n is square of an odd integer;

(ii) $\beta = 2$ and $n \equiv 0 \pmod 4$ or $(n+1)/2$ is square of an odd integer;

(iii) $\beta = 4$ and $n+3$ is square of an even integer or $(n+1)/2$ is square of the rational part of $(1+\sqrt{2})^{2t+1}$ for some positive integer t.

Theorem 9. (A. Jindal, S. Laishram), 2023

Let $n \geq 1$ be an integer and $\gamma \in \{-6,-1,0,1,2,3\}$ be such that when $\gamma = 0$, then the highest power of 2 dividing n does not exceed $n^{1/2}$. The associated Galois group of $L_n^{(n+\gamma)}(x)$ is A_n if and only if one of the following conditions is satisfied:

(i) $\gamma = -6$ and $(2n-5)/3$ is the rational part of $(1+\sqrt{2})^{4t}$ for some positive integer t;

(ii) $\gamma = -1$ and n is square of an odd integer;

(iii) $\gamma = 0$ and $n \equiv 0 \pmod 2$;

(iv) $\gamma = 1$ and $n+1$ is twice a square;

(v) $\gamma = 2$ and $n+1$ is the square of the rational part of $(1+\sqrt{2})^{2t+1}$ for some positive integer t;

(vi) $\gamma = 3$ and n is square of an even integer.

Refrences

G. Dumas, *Sur quelques cas d'irréductibilité des polynômes á coefficients rationnels*, Journal de Math. Pure et Appl. 2 (1906) 191-258.

I. Schur, *Einige Sätze über Primzahlen mit Anwendungen auf Irreduzibilitätsfragen, I*, Sitzungsber. Preuss. Akad. Wiss. Berlin Phys.-Math. Kl. 14 (1929) 125-136.

I. Schur, *Einige Sätze über Primzahlen mit Anwendungen auf Irreduzibilitätsfragen, II*, Sitzungsber. Preuss. Akad. Wiss. Berlin Phys.-Math. Kl. 14 (1929) 370-391.

I. Schur, *Gleichungen ohne Affekt*, Sitzungsberichte der Preussischen Akademie der Wissenschaften, Physikalisch-Mathematische Klasse (1930) 443-449.

I. Schur, *Affektlose Gleichungen in der Theorie der Laguerreschen und Hermiteschen Polynome*, Journal für die reine und angewandte Mathematik 165 (1931) 52-58.

Refrences

- R. F. Coleman, *On the Galois groups of the exponential Taylor polynomials*, L'Enseignement Math. 33 (1987) 183-189.
- F. Hajir, *Some $\tilde{A}_n$-extensions obtained from generalized Laguerre polynomials*, J. Number Theory 50 (1995) 206-212.
- F. Hajir, *Algebraic properties of a family of generalized Laguerre polynomials*, Canad. J. Math. 61 (2009) 583-603.
- M. Filaseta and O. Trifonov, *The Irreducibility of the Bessel polynomials*, J. Reine Angew. Math. 550 (2002) 125-140.
- M. Filaseta, T. Kidd and O. Trifonov, *Laguerre polynomials with Galois group A_m for each m*, J. Number Theory 132 (2012) 776-805.

Refrences

A. Jindal, S. Laishram and R. Sarma, *Irreducibility and Galois groups of Generalised Laguerre Polynomials* $L_n^{(-1-n-r)}(x)$, J. Number Theory 183 (2018) 388-406.

A. Jindal and S. Laishram, *Families of Laguerre polynomials with Alternating group as Galois group*, J. Number Theory 241 (2022) 387-429.

A. Jindal and S. Laishram, *Families of Laguerre polynomials with Alternating group as Galois group II*, preprint.

A. Jindal, S. G. Nair and T. N. Shorey, *Extension of Irreducibility results on Generalised Laguerre Polynomials* $L_n^{(-1-n-s)}(x)$, preprint.

Refrences

- D. H. Lehmer, *On a problem of Störmer*, Illinois J. Math. 8 (1964) 57-79.
- F. Luca and F. Najman, *On the largest prime factor of* $x^2 - 1$, Mathematics of Computation 80 (2011) 429-435.
- S. G. Nair and T. N. Shorey, *Irreducibility of Laguerre Polynomial* $L_n^{(-1-n-r)}(x)$, Indagationes Mathematicae 26 (2015) 615-625.
- E.A Sell, *On a certain family of generalized Laguerre polynomials*, J. Number Theory 107 (2004) 266-281.
- T.N. Shorey and S. B. Sinha, *Extension of Laguerre Polynomials with negative arguments*, Indag. Math. 33 (2022) 801–815.

Thank You