

Diophantine equations and Prouhet-Tarry-Escott sets

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Main question

Let $a_0, a_1, \dots, a_k \in \mathbb{Q}$, distinct, $a_0 \neq 0$. Put

$$f(x) = a_0(x - a_1) \cdots (x - a_k).$$

Let $b_0, b_1, \dots, b_\ell \in \mathbb{Q}$, distinct, $b_0 \neq 0$.

Put

$$g(y) = b_0(y - b_1) \cdots (y - b_\ell).$$

For which f, g has equation

$$f(x) = g(y)$$

infinitely many solutions?

Main question (2)

Let $a_0, a_1, \dots, a_k \in \mathbb{Q}$, $a_0 \neq 0$. Put $f(x) = a_0(x - a_1) \cdots (x - a_k)$.

Let $b_0, b_1, \dots, b_\ell \in \mathbb{Q}$, $b_0 \neq 0$. Put $g(y) = b_0(y - b_1) \cdots (y - b_\ell)$.

Equation $f(x) = g(y)$ has infinitely many rational solutions *with a bounded denominator* if there is a $\Delta \in \mathbb{Z}$ such that $f(x) = g(y)$ has infinitely many solutions with $(\Delta x, \Delta y) \in \mathbb{Z}^2$.

For which f, g has the equation $f(x) = g(y)$ infinitely many solutions $(x, y) \in \mathbb{Q}^2$ with a bounded denominator?

There are many results in the literature on this equation.

Example 1

1. Let

$$f(x) = (x - 6)(x + 6), \quad g(y) = (y - 1)(y - 4)(y - 9).$$

Then $f(x) = g(y)$ has solution

$$(x, y) = (q(q^2 - 7), q^2) \quad \text{for every } q \in \mathbb{Q}.$$

Indeed

$$(q^3 - 7q - 6)(q^3 - 7q + 6) = (q^2 - 1)(q^2 - 4)(q^2 - 9)$$

since

$$q^3 - 7q + 6 = (q - 1)(q - 2)(q + 3).$$

Example 2

Consider the Pell equation $x^2 = 2y^2 - 1$. It has solutions $(X_n, Y_n)_{n=1}^{\infty}$ given by $(X_1, Y_1) = (1, 1)$, $(X_2, Y_2) = (7, 5)$ and

$$X_{n+1} = 6X_n - X_{n-1}, \quad Y_{n+1} = 6Y_n - Y_{n-1} \quad (n \in \mathbb{Z}).$$

Let

$$f(x) = (x-7)(x-1)(x+1)(x+7), \quad g(y) = 4(y-5)(y-1)(y+1)(y+5).$$

Then $f(x) = g(y)$ has solutions

$$(x, y) = (X_n, Y_n) \quad \text{for every } n \in \mathbb{Z}.$$

Indeed $X_n^2 - X_m^2 = 2(Y_n^2 - Y_m^2)$.

Old results (1). Some results on the equation
 $x(x+1)\cdots(x+k-1) = y(y+1)\cdots(y+\ell-1)$
in $x, y \in \mathbb{Z}_{>0}$.

Mordell (1963): Solutions for $(k, \ell) = (2, 3)$ are $(x, y) = (2, 1), (14, 5)$.

Boyd and Kisilevsky (1972): The solutions for $(k, \ell) = (3, 4)$ are
 $(2, 1), (4, 2), (55, 19)$.

Saradha and Shorey (1990): The only solution with $\ell = 2k$ is given by
 $(k, \ell, x, y) = (3, 6, 8, 1)$.

Mignotte, Saradha, Shorey (1996): No solutions for $\ell/k \in \{3, 4, 5, 6\}$.

Hajdu and Pintér (2000): The solution for $(k, \ell) = (4, 6)$ is $(7, 2)$.

Old results (2). Some results on equation $\binom{m}{k} = \binom{n}{\ell}$.

Without loss of generality we assume $k < \ell$, $k \leq m/2$, $\ell \leq n/2$.

Mordell (1963): no solutions if $(k, \ell) = (3, 4)$.

Avanesov (1966): $(k, \ell) = (2, 3)$ admits only solutions $(m, n) = (16, 10), (56, 22), (120, 36)$.

Pintér (1995), De Weger (1996): The only solution for $(k, \ell) = (2, 4)$ is $(m, n) = (21, 10)$.

Stroecker and De Weger (1999): The only solution for $(k, \ell) \in \{(2, 6), (2, 8), (3, 6), (4, 6), (4, 8)\}$ is $(k, \ell, m, n) = (2, 8, 221, 17)$.

Bugeaud, Mignotte, Siksek, Stoll and Tengely (2008): no solutions if $(k, \ell) = (2, 5)$.

Blokhuis, Brouwer, De Weger (2017): All solutions are known for $m \leq 10^6$ and for $\binom{m}{k} = \binom{n}{\ell} \leq 10^{60}$.

Old results (3). Some results on

$$x(x+d_1) \cdots (x+(k-1)d_1) = y(y+d_2) \cdots (y+(\ell-1)d_2).$$

Saradha, Shorey and Tijdeman (1995): Case $d_1 = 1$, $d_2 > 1$, $\ell/k \in \mathbb{Z}$.

Beukers, Shorey and Tijdeman (1999). Only finitely many positive integral solutions x, y except when $(k, \ell) = (2, 4)$ and $d_1 = 2d_2^2$.

Then

$$(y^2 + 3d_2y)(y^2 + 3d_2y + 2d_2^2) = y(y + d_2)(y + 2d_2)(y + 3d_2).$$

Related result: Let $f_{k,m}(x) = \frac{x(x+1)\cdots(x+k-2)((m-2)x+k+2-m)}{k!}$.

(So-called figurate number)

Hajdu, Pintér, Tengely and Varga (2014): Various finiteness results on the equation

$$f_{k,m}(x) = f_{\ell,n}(y)$$

Theorem

Let $f(x), g(x) \in \mathbb{Q}[x]$ have only simple rational roots.

Suppose $\deg(f) \leq \deg(g)$.

Let $f(x) = g(y)$ have infinitely many solutions $(x, y) \in \mathbb{Q}^2$ with a bounded denominator. Then

$$\deg(f) \mid 2 \deg(g).$$

We have seen an example with $\deg(f) = 2, \deg(g) = 3$.

Thus a) with $\deg(f) \mid \deg(g)$ is wrong.

Remaining question:

Which pairs $(\deg(f), \deg(g))$ are possible?

Some definitions

We call polynomials $f, g \in \mathbb{Q}[x]$ *similar* if there exist $a, b \in \mathbb{Q}$, $a \neq 0$ such that $f(x) = g(ax + b)$. Notation $f \simeq g$.

This induces an equivalence relation in $\mathbb{Q}[x]$.

If f has only simple rational roots, then g has only simple rational roots. In every equivalence class there is a polynomial with integer roots and sum of roots equal to 0.

We call $f(x) \in \mathbb{Q}[x]$ *decomposable* if there exist $G(x), H(x) \in \mathbb{Q}[x]$ with $\deg(G) > 1$, $\deg(H) > 1$ such that $f = G(H)$,

Let b be a non-zero rational number and m be a positive integer. Then the m -th *Dickson polynomial* is defined by

$$D_m(x, b) := \sum_{i=0}^{\lfloor m/2 \rfloor} d_{m,i} x^{m-2i} \quad \text{where } d_{m,i} = \frac{m}{m-i} \binom{m-i}{i} (-b)^i.$$

Auxiliary results (1)

We denote the discriminant of polynomial P by $\text{disc}(P)$.

Theorem (Davenport, Lewis, Schinzel, 1961)

Let $F(x) \in \mathbb{Z}[x]$ be of degree $m > 1$ and $G(y) \in \mathbb{Z}[x]$ of degree $n > 1$.
Let

$$D(z) = \text{disc}(F(x) + z), \quad E(z) = \text{disc}(G(y) + z).$$

Suppose there are at least $\lceil \frac{1}{2}m \rceil$ roots of $D(z) = 0$ for which $E(z) \neq 0$.
Then $F(x) - G(y)$ is irreducible over the complex field. Further, the genus of the equation $F(x) - G(y) = 0$ is positive except possibly when $n = 2$ or $m = n = 3$.

Theorem (Siegel, 1929)

The number of integral points on an irreducible algebraic curve of positive genus is finite.

Auxiliary results (2)

Theorem (Bilu, Tichy, 2000)

Let $f(x), g(x) \in \mathbb{Q}[x]$ be non-constant polynomials. Then the following two statements are equivalent.

- (I) The equation $f(x) = g(y)$ has infinitely many rational solutions x, y with a bounded denominator.
- (II) We have $f = \varphi(F(\kappa))$ and $g = \varphi(G(\lambda))$, where $\kappa(x), \lambda(x) \in \mathbb{Q}[x]$ are linear polynomials, $\varphi(x) \in \mathbb{Q}[x]$, and $F(x), G(x)$ form a standard pair over \mathbb{Q} such that the equation $F(x) = G(y)$ has infinitely many rational solutions with a bounded denominator.

Note that $F(\kappa) \sim F, G(\lambda) \sim G$. (We often identify them.)

(II) implies (I) is trivial.

Notation: $k = \deg(f), \ell = \deg(g), m = \deg(F), n = \deg(G), s = \deg(\varphi)$. Therefore $k = ms, \ell = ns$.

There are five kinds of standard pairs.

Standard pairs

Kind	Standard pair (F, G unordered)	Parameter restrictions
First	$(x^q, ax^p v(x)^q)$	$0 \leq p < q, (p, q) = 1,$ $p + \deg(v) > 0$
Second	$(x^2, (ax^2 + b)v(x)^2)$	-
Third	$(D_m(x, a^n), D_n(x, a^m))$	$\gcd(m, n) = 1$
Fourth	$(a^{-m/2} D_m(x, a), -b^{-n/2} D_n(x, b))$	$\gcd(m, n) = 2$
Fifth	$((ax^2 - 1)^3, 3x^4 - 4x^3)$	-

Standard pairs. Here

a, b are non-zero rational numbers,

m, n, q are positive integers,

p is a non-negative integer,

$v(x) \in \mathbb{Q}[x]$ is a non-zero, but possibly constant polynomial.

Earlier applications of the Bilu-Tichy theorem

Consider the equation $f(x) := (x + d_1 d) \cdots (x + d_k d) = g(y)$
where $d, k, K, d_1, d_2, \dots, d_k \in \mathbb{Z}$ with $0 \leq d_1 < d_2 < \cdots < d_k < K$,
 $k > 2$, $g(y) \in \mathbb{Q}[y]$ of degree $\ell \geq 2$.

Kulkarni and Sury (2003): The number of solutions of the equation is finite under the assumption that $d = 1$, $k = K$, $\ell > 2$, provided that g does not belong to three explicitly given classes in which there can be infinitely many solutions.

Hajdu, Papp and Tijdeman (to appear): The number of solutions of the equation is finite under the assumption that $K - k \leq cK^{2/3}$ with c an explicit constant, provided that g does not belong to two explicitly given classes in which there can be infinitely many solutions.

Standard pairs of the fifth kind

Theorem

Let $f(x), g(x) \in \mathbb{Q}[x]$ have only simple rational roots.

Suppose $\deg(f) \leq \deg(g)$.

Let $f(x) = g(y)$ have infinitely many solutions $(x, y) \in \mathbb{Q}^2$ with a bounded denominator. Then $\deg(f) \mid 2 \deg(g)$.

A standard pair of the fifth kind is $(F, G) = ((\alpha x^2 - 1)^3, 3x^4 - 4x^3)$.

Suppose f has only simple rational roots.

Then f' has only simple real roots too.

Since $f = \varphi(F)$ we have $f' = \varphi'(F) \cdot F'$.

Therefore F' has only simple real roots.

This is not the case for standard pairs of the fifth kind.

Thus we can exclude the standard pairs of the fifth kind.

Restrictions if f has only simple rational roots

Theorem

Assume that with some rational numbers u, b with $ub \neq 0$ we have

$$D_n(x, b) + u = (x + w_1) \dots (x + w_n), \quad (1)$$

where $D_n(x, b)$ is the n -th Dickson polynomial with parameter b and $w_1, \dots, w_n \in \mathbb{Q}$ are distinct. Then $n \in \{1, 2, 3, 4, 6\}$.

Theorem

Let $n \in \{3, 4, 6\}$. For any $w_1, w_2 \in \mathbb{Q}$ we can define $w_3, \dots, w_n, b, u \in \mathbb{Q}$ such that (1) holds. On the other hand, this provides the only solutions of equation (1).

Case $n \in \{1, 2\}$ is simple.

Complicated proofs, case by case.

The case $n = 6$

$$\text{Let } D_n(x, b) + u = (x + w_1) \dots (x + w_n), \quad (1)$$

where $D_n(x, b)$ is the n -th Dickson polynomial with parameter b and $w_1, \dots, w_n \in \mathbb{Q}$ are distinct. Then $N \in \{1, 2, 3, 4, 6\}$.

Let $n \in \{3, 4, 6\}$. For any $w_1, w_2 \in \mathbb{Q}$ we can choose $w_3, \dots, w_n, b, u \in \mathbb{Q}$ such that (1) holds. On the other hand, this provides the only solutions of equation (1).

$n = 6$: Then the only solution of (1) is given by

$$w_3 := w_1 + w_2, \quad w_4 := -w_1, \quad w_5 := -w_2, \quad w_6 := -w_3.$$

$$W := w_1^2 + w_1 w_2 + w_2^2,$$

$$b := \frac{W}{3}, \quad u := \frac{2W^3}{27} - w_1^2 w_2^2 (w_1 + w_2)^2. \quad (2)$$

Standard pairs of the fourth kind

Fourth kind: $(F, G) = (a^{-m/2}D_m(x, a), -b^{-n/2}D_n(x, b))$, $\gcd(m, n) = 2$.

$$\text{Let } D_n(x, b) + u = (x + w_1) \dots (x + w_n), \quad (1)$$

where $D_n(x, b)$ is the n -th Dickson polynomial with parameter b .
Then $n \in \{1, 2, 3, 4, 6\}$.

Suppose $n > m > 2$. Then the only possibility is $(m, n) = (4, 6)$.
Therefore the equation

$$a^{-2}D_4(x, a) = -b^{-3}D_6(y, b) \quad (3)$$

has infinitely many rational solutions x, y with bounded denominator.

We have $n = 6$: $W = w_1^2 + w_1 w_2 + w_2^2$, $b = \frac{W}{3} > 0$.

D_4, D_6 are polynomials of even degree with opposite signs.

The equation can have only finitely many such solutions. Contradiction.

Standard pairs of the third kind (1)

Third kind: $(F, G) = (D_m(x, a^n), D_n(x, a^m))$, $\gcd(m, n) = 1$.

Suppose $n > m > 2$. Then the only possibility is $(m, n) = (3, 4)$.
Therefore the equation

$$D_3(x, a^4) = D_4(y, b^3) \tag{4}$$

has infinitely many rational solutions x, y with bounded denominator.
It follows that for some a_1, a_2, b_1, b_2 the equation

$$(x - a_1)(x - a_2)(x + a_1 + a_2) = (y^2 - b_1^2)(y^2 - b_2^2)$$

has infinitely many such solutions.

Standard pairs of the third kind (2)

Third kind: $(F, G) = (D_m(x, \alpha^n), D_n(x, \alpha^m))$, $\gcd(m, n) = 1$.

Suppose $n > m > 2$. Then the only possibility is $(m, n) = (3, 4)$.
Therefore the equation

$$D_3(x, a^4) = D_4(y, b^3) \quad (5)$$

has infinitely many rational solutions x, y with bounded denominator.
It follows that for some a_1, a_2, b_1, b_2 the equation

$$(x - a_1)(x - a_2)(x + a_1 + a_2) = (y^2 - b_1^2)(y^2 - b_2^2)$$

has infinitely many such solutions.

Lemma. *Let a, b, c be rational numbers such that $3a^2 + b^2 = c^2$.
Then there exist rational numbers u, v, w such that*

$$a^2 = w^2(2uv)^2, \quad b^2 = w^2(3u^2 - v^2)^2, \quad c^2 = w^2(3u^2 + v^2)^2$$

It follows that $\sqrt{27} \in \mathbb{Q}$. Contradiction.

The remaining cases are cases with standard pairs of the first or second kind and standard pairs of the third or fourth kind with $m = \deg(F) \leq 2$.

We have $f = \varphi(F)$. Let $\varphi(x) = p_0(x - p_1) \cdots (x - p_s)$.

Then p_1, p_2, \dots, p_s distinct and $f(x) = (F(x) - p_1) \cdots (F(x) - p_s)$ has only single rational roots with a bounded denominator.

Thus, each of $F(x) - p_1, F(x) - p_2, \dots, F(x) - p_s$ has only single rational roots.

Proof of the main theorem

Theorem:

Let $f(x), g(x) \in \mathbb{Q}[x]$ have only simple rational roots.

Suppose $\deg(f) \leq \deg(g)$.

Let $f(x) = g(y)$ have infinitely many solutions $(x, y) \in \mathbb{Q}^2$ with a bounded denominator. Then $\deg(f) \mid 2 \deg(g)$.

Proof. $m = \deg(F) \leq 2$ or standard pairs of the first or second kind.

Bilu-Tichy: The equation $F(x) = G(y)$ has infinitely many rational solutions with a bounded denominator.

First kind: $F(x) = x^q$, second kind: $F(x) = x^2$.

"Thus, each of $F(x) - p_1, F(x) - p_2, \dots, F(x) - p_s$ has only single rational roots."

It follows that $x^q - p_1$ has only simple rational roots. Thus $q \leq 2$.

Thus $\deg(F) \leq 2$, hence $\deg(F) \mid 2 \deg(G)$ and also $\deg(f) \mid 2 \deg(g)$.

PTE-sets

p_1, p_2, \dots, p_s are distinct and $f(x) = (F(x) - p_1) \cdots (F(x) - p_s)$ has only single rational roots with a bounded denominator.

Set

$$F(x) - p_i = p_{i0}(x - p_{i1}) \cdots (x - p_{im}) \quad (i = 1, 2, \dots, s).$$

By the rules of Newton-Girard,

$\sum_j p_{ij}, \sum_j p_{ij}^2, \dots, \sum_j p_{ij}^{m-1}$ are independent of i . We call such sets $F(x) - p_i$ ($i = 1, 2, \dots, s$) with only simple rational roots PTE-sets.

$s = 2$, only integer roots: 'ideal Prouhet-Tarry-Escott pairs'.

Known to exist for $m \leq 12, m \neq 11$.

Theorem

For $m = \deg(F) \in \{2, 3, 4, 6\}$ there exist PTE-sets with any $s \in \mathbb{Z}_{>0}$.

For $m = 2$ this is trivial.

PTE-sets are useful to construct equations $f(x) = g(y)$ with infinitely many integer solutions with f, g having only simple integral roots.

Lemma

Let N be the product of r primes of the form $\equiv 1 \pmod{4}$. Then N can be written as the sum of two squares of positive integers in 2^r ways.

We take ($r = 3$)

$$5 \cdot 13 \cdot 17 = 1105 = x^2 + y^2 \text{ for } (x, y) = (33, 4), (32, 9), (31, 12), (24, 23).$$

Hence $G(x) = x^4 - 1105x^2$ has simple rational roots when 17424, 82944, 138384 or 304704 is added, since the corresponding polynomials equal $(x^2 - 33^2)(x^2 - 4^2)$,

$$(x^2 - 32^2)(x^2 - 9^2), (x^2 - 31^2)(x^2 - 12^2), (x^2 - 24^2)(x^2 - 23^2).$$

A PTE-quadruple of degree 4.

Case $n = 4$

$5 \cdot 13 \cdot 17 = 1105 = x^2 + y^2$ for $(x, y) = (33, 4), (32, 9), (31, 12), (24, 23)$.

Hence $G(x) = x^4 - 1105x^2$ has simple rational roots when 17424, 82944, 138384 or 304704 is added.

$(x^2 - 33^2)(x^2 - 4^2), (x^2 - 32^2)(x^2 - 9^2), (x^2 - 31^2)(x^2 - 12^2), (x^2 - 24^2)(x^2 - 23^2)$, respectively

Put $F(x) = x$,

$$f(x) = \varphi(x) = (x + 17424)(x + 82944)(x + 138384)(x + 304704).$$

Then $g(y) = \varphi(G(y))$ is given by

$$(y^2 - 33^2)(y^2 - 4^2)(y^2 - 32^2)(y^2 - 9^2)(y^2 - 31^2)(y^2 - 12^2)(y^2 - 24^2)(y^2 - 23^2)$$

For every integer X we obtain a solution $(x, y) = (G(X), X)$.

Here $k = 4, \ell = 16, m = 1, n = 4, s = 4$.

PTE's of degree 6

Lemma. *Let N be the product of r primes of the form $\equiv 1 \pmod{6}$. Then N can be written as $x^2 + xy + y^2$ for positive integers x, y in 2^r ways.*

We take $(r = 3) : 7 \cdot 13 \cdot 19 =$

$$1729 = x^2 + xy + y^2 \text{ for } (x, y) = (40, 3), (37, 8), (32, 15), (25, 23).$$

Hence $G(x) = x^6 - 2 \cdot 1729x^4 + 1729^2x^2$ has simple rational roots when 26625600, 177422400, 508953600 or 761760000 is subtracted, since the corresponding polynomials equal

$$(x^2 - 40^2)(x^2 - 3^2)(x^2 - 43^2), (x^2 - 37^2)(x^2 - 8^2)(x^2 - 45^2),$$

$$(x^2 - 32^2)(x^2 - 15^2)(x^2 - 47^2), (x^2 - 25^2)(x^2 - 23^2)(x^2 - 48^2).$$

A PTE-quadruple of degree 6.

Case $n = 6$

$G(x) = x^6 - 2 \cdot 1729x^4 + 1729^2x^2$ has simple rational roots when 26625600, 177422400, 508953600 or 761760000 is subtracted, since the corresponding polynomials equal

$$(x^2 - 40^2)(x^2 - 3^2)(x^2 - 43^2), (x^2 - 37^2)(x^2 - 8^2)(x^2 - 45^2), \\ (x^2 - 32^2)(x^2 - 15^2)(x^2 - 47^2), (x^2 - 25^2)(x^2 - 23^2)(x^2 - 48^2).$$

Put $F(x) = x$, $f(x) = \varphi(x) =$

$$(x - 26625600)(x - 177422400)(x - 508953600)(x - 761760000).$$

Then $g(y) = \varphi(G(y))$ is given by

$$\prod_{t \in T} (x^2 - t^2) \text{ for } T = \{40, 3, 43, 37, 8, 45, 32, 15, 47, 25, 23, 48\}.$$

For every integer X we obtain a solution $(x, y) = (G(X), X)$.

Here $k = 4, \ell = 24, m = 1, n = 6, s = 4$.

The polynomial $P(x) = x^3 - 1729^2x$ has simple integer roots when one from

$$0, \pm 728932560, \pm 1678772880, \pm 1878480960, \pm 286101600$$

is added. Namely, we get the polynomials $(x - 1729)x(x + 1729)$,
 $(x \pm 1840)(x \mp 249)(x \mp 1591)$, $(x \pm 1961)(x \mp 656)(x \mp 1305)$,
 $(x \pm 1984)(x \mp 1185)(x \mp 799)$, $(x \pm 1775)(x \mp 96)(x \mp 1679)$.

Case $n = 3$

The polynomial $G(x) = x^3 - 1729^2x$ has simple integer roots when one from

$0, \pm 728932560, \pm 1678772880, \pm 1878480960, \pm 286101600$

is added. Namely, we get the polynomials $(x - 1729)x(x + 1729)$,

$$(x \pm 1840)(x \mp 249)(x \mp 1591), (x \pm 1961)(x \mp 656)(x \mp 1305),$$

$$(x \pm 1984)(x \mp 1185)(x \mp 799), (x \pm 1775)(x \mp 96)(x \mp 1679).$$

Put $F(x) = x$, $f(x) = \varphi(x) = x(x^2 - 728932560^2) \times$
 $(x^2 - 1678772880^2)(x^2 - 1878480960^2)(x^2 - 286101600^2)$, and
 $g(y) = y \prod_{a \in T} (y^2 - a^2)$ with $T =$
 $\{1729, 1840, 249, 1591, 1961, 656, 1305, 1984, 1185, 799, 1775, 96, 1679\}$

For every integer X we obtain a solution $(x, y) = (G(X), X)$.

Here $k = 9, \ell = 27, m = 1, n = 3, s = 9$.

Examples with $F(x) = x^2$

Kind	Standard pair (F, G unordered)	Parameter restrictions
First	$(x^q, ax^p v(x)^q)$	$0 \leq p < q, (p, q) = 1,$ $p + \deg(v) > 0$
Second	$(x^2, (ax^2 + b)v(x)^2)$	-

We know $q \leq 2$.

Remaining possibilities with $F(x) = x^2$, i.e. $m = q = 2$:

1. First kind: $p = 0, G(y) = av(y)^2,$
2. First kind: $p = 1, G(y) = ayv(y)^2,$
3. Second kind: $G(y) = (ay^2 + b)v(y)^2$ with $ab \neq 0$.

We give examples for each case.

Obviously every example with $F(x) = x^2$ leads to an example with $F(x) = x$.

Example with $F(x) = x^2$, $G(y)$ is of the form $v(y)^2$

It is known that the sets

$$T_1 := \{\pm 22, \pm 61, \pm 86, \pm 127, \pm 140, \pm 151\},$$

$$T_2 := \{\pm 35, \pm 47, \pm 94, \pm 121, \pm 146, \pm 148\}$$

form an ideal $\text{PTE}_{2,12}$ pair. Let $v(y) = (\prod_{t \in T_1} (y - t) + \prod_{t \in T_2} (y - t))/2$ and $v_0 = (\prod_{t \in T_1} t - \prod_{t \in T_2} t)/2$. Then

$$g(y) := \prod_{t \in T_1 \cup T_2} (y - t) = (v(y) + v_0)(v(y) - v_0) = v^2(y) - v_0^2.$$

We define $f(x) = x^2 - v_0^2 = (x - v_0)(x + v_0)$. Thus f, g both have simple rational roots and the equation $f(x) = g(y)$ has solutions $(x, y) = (\pm v(X), X)$ for every $X \in \mathbb{Q}$. We have $\varphi(x) = x - v_0^2$, $F(x) = f(x) + v_0^2 = x^2$, $G(y) = g(y) + v_0^2 = v(y)^2$, $\deg(v) = 12$. Thus $k = 2, \ell = 24, m = 2, n = 24, s = 1$.

Example with $F(x) = x^2$, $G(y)$ is of the form $yv(y)^2$

It is known that the symmetric sets

$$T_3 := \{-98, -82, -58, -34, 13, 16, 69, 75, 99\} \text{ and } T_4 := \{t \in T_3 : -t\}$$

form an ideal PTE pair. Put $g(y) = \prod_{t \in T_3} (y - t^2)$, $A = \prod_{t \in T_3} t$ and $yT(y) = \prod_{t \in T_3} (y - t) + A$. Then

$$g(y^2) := \prod_{t \in T_3} (y - t) \cdot \prod_{t \in T_4} (y - t) = (yT(y) - A)(yT(y) + A).$$

Observe that $yT(y)$ is an odd function, so $T(y)$ is an even function (i.e., the coefficients of y^i with i odd are 0 in T). This yields that

$T(y) = v(y^2)$ with $v(y) \in \mathbb{Q}[y]$ and therefore $g(y^2) = y^2 v(y^2)^2 - A^2$.

So letting $f(x) = (x - A)(x + A)$, we see that the equation $f(x) = g(y)$ has solutions $(x, y) = (Xv(X^2), X)$ for every $X \in \mathbb{Q}$. We have

$\varphi(x) = x - A^2$, $F(x) = x^2$, $G(y) = yv(y)^2$, $\deg(v) = 4$.

Here $k = 2$, $\ell = 9$, $m = 2$, $n = 9$, $s = 1$.

Example with $F(x) = x^2$, $G(y)$ is of the form $y^2 + b$,
 $b \neq 0$, ($v(x) = 1$ for all x)

Suppose that the equation $x^2 = ay^2 + b$ with $a, b \in \mathbb{Z}$, $ab \neq 0$ has solutions $(X_i, Y_i)_{i=1}^{\infty} \in \mathbb{Z}^2$. Let $s \geq 1$, and put

$$F(x) = x^2, \quad G(y) = ay^2 + b, \quad \varphi(x) = \prod_{i=1}^s (x - X_i^2).$$

Then we have

$$f(x) = \prod_{i=1}^s (x^2 - X_i^2), \quad g(y) = \prod_{i=1}^s (ay^2 + b - X_i^2) = a^s \prod_{i=1}^s (y^2 - Y_i^2).$$

So $f(x)$ and $g(y)$ both have simple rational roots.

Further, the equation $f(x) = g(y)$ has solutions (X_i, Y_i) for $i = 1, 2, \dots$

Here $k = 2s, \ell = 2s, m = 2, n = 2, s$ is arbitrary.

Application. *For every positive integer N there exist only finitely many pairs of disjoint blocks A and B of size at most N with the property that for some k, ℓ with $1 \leq k < \ell$ and $k \nmid 2\ell$, there exist distinct elements $a_1, \dots, a_k \in A$ and distinct elements $b_1, \dots, b_\ell \in B$ such that*

$$a_1 \cdots a_k = b_1 \cdots b_\ell.$$

Remark. We have corresponding results on the equation $f(x) = g(y)$ in case f has only simple rational roots and we only know $g(y) \in \mathbb{Q}[y]$.

Open problems

1. Which pairs $(\deg(f), \deg(g))$ are possible such that f, g both have only simple rational roots and $f(x) = g(y)$ has infinitely many integral solutions?

In particular

2. What is the maximum of $\deg(g)/\deg(f)$?

Replace $F(x) = x^2$ by $F(x) = x$.

Then ideal PTE pair of degree 12 gives quotient 24.

3. What is the maximum of $\deg(g)/\deg(f)$ if $\deg(f)$ is not a divisor of $\deg(g)$?

Ideal PTE pair of degree 9 gives quotient $9/2$.

4. What is the maximal degree of v if $F(x) = x^2$, $G(y)$ is of the form $(ay^2 + b)v(y)^2$, $ab \neq 0$?

(Standard pair of the second kind. Up to now only v is constant.)