On (non)primitive Diophantine Approximation

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Story No. 1

(Non)primitive points

An exercise for students

$$n = \text{least common multiple}\,[1, 2, 3, ..., k]$$

Prime Number Theorem $\log n \sim k, \quad k \to \infty$

Corollary

Among $k - 1 \sim \log n$ numbers $2, 3, ..., k$ there is no numbers coprime to n.

Jacobsthal's function

Jacobsthal's function $j(n)$ represents the smallest number Q such that every sequence of Q consecutive integers contains an integer coprime to n.

- Jacobsthal's conjecture: $j(n) = O\left(\left(\frac{\log n}{\log\log n}\right)^2\right)$
- Lower bound Ford, Green, Konyagin, Tao (2015):

$$j(n) \gg f(n) \cdot \frac{\log n \, \log\log n \, \log\log\log\log n}{(\log\log\log n)^2}, \quad f(n) \to \infty, \quad \text{for certain } n.$$

- Upper bound: $j(n) = O((\log n)^2)$

Iwaniec, H.: On the problem of Jacobsthal
Demonstratio Mathematica 11(1), 225-231 (1978)

- Some history:

P. Erdős : On the integers relatively prime to n and on a number-theoretic function considered by Jacobsthal,
Math. Scand. 11(1962) 163-170

Two-dimensional version of Jacobsthal's problem

Erdős: On an elementary problem in number theory, Canadian Math. Bull. 1, (1958), 5-8.

1) For every $\varepsilon > 0$ there exist arbitrarily large positive integer x and $y \geq x$ such that

$$\text{g.c.d.}(x+i, y+j) > 1$$

for all pairs i, j with

$$0 \leq i, j \leq (1-\varepsilon)\left(\frac{\log x}{\log\log x}\right)^{1/2}.$$

2) For a certain positive constant c for any positive integers $x \leq y$ there exist a pair of integers i, j with

$$0 \leq i, j \leq c\,\frac{\log x}{\log\log x}$$

such that

$$\text{g.c.d.}(x+i, y+j) = 1.$$

Story No. 2

Inhomogeneous approximation to one number

What is inhomogeneous approximation?

$$\theta, \eta \in \mathbb{R}$$

$$x, y \in \mathbb{Z}$$

homogeneous approximation : $|\theta x + y|$ is small

inhomogeneous approximation : $|\theta x - \eta + y|$ is small

Hurwitz and Khintchine

• Hurwitz (late XIX century) Let $\theta \in \mathbb{R} \setminus \mathbb{Q}, \eta \in \mathbb{R}$
then $\exists$ inf. many $q \in \mathbb{Z}_+, r \in \mathbb{Z}$ such that

$$|q\theta - r| = ||q\theta|| \leq \frac{1}{\sqrt{5} \cdot q}$$

• Khintchine (1935) $\varepsilon > 0$. Let $\theta \in \mathbb{R} \setminus \mathbb{Q}, \eta \in \mathbb{R}$
then $\exists$ inf. many $q \in \mathbb{Z}_+, r \in \mathbb{Z}$ such that

$$|q\theta - \eta - r| = ||q\theta - \eta|| \leq \frac{1 + \varepsilon}{\sqrt{5} \cdot q}$$

• If $\eta \neq m\alpha + n$, $m, n \in \mathbb{Z}$ the inhomogeneous result may be improved:
Cassels (1954)

$$\liminf_{q \to \infty} q \, ||q\theta - \eta|| \leq \frac{27}{28\sqrt{7}}$$

Khintchine's Palermo paper (1926):
Über eine Klasse linearer diophantischer Approximationen

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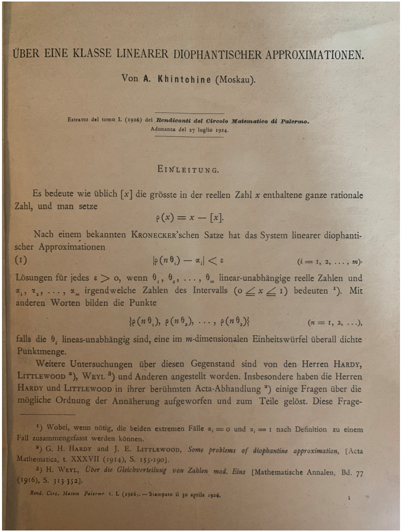

ÜBER EINE KLASSE LINEARER DIOPHANTISCHER APPROXIMATIONEN.

Von A. Khintchine (Moskau).

Estratto del tomo L (1926) dei *Rendiconti del Circolo Matematico di Palermo.*
Adunanza del 27 luglio 1924.

EINLEITUNG.

Es bedeute wie üblich $[x]$ die grösste in der reellen Zahl x enthaltene ganze rationale Zahl, und man setze

$$\rho(x) = x - [x].$$

Nach einem bekannten KRONECKER'schen Satze hat das System linearer diophantischer Approximationen

$$(1) \qquad |\rho(n\theta_i) - \alpha_i| < \varepsilon \qquad (i = 1, 2, \ldots, m).$$

Lösungen für jedes $\varepsilon > 0$, wenn $\theta_1, \theta_2, \ldots, \theta_m$ linear-unabhängige reelle Zahlen und $\alpha_1, \alpha_2, \ldots, \alpha_m$ irgendwelche Zahlen des Intervalls $(0 \leqq x \leqq 1)$ bedeuten [1]). Mit anderen Worten bilden die Punkte

$$\{\rho(n\theta_1), \rho(n\theta_2), \ldots, \rho(n\theta_2)\} \qquad (n = 1, 2, \ldots),$$

falls die θ_i lineas-unabhängig sind, eine im m-dimensionalen Einheitswürfel überall dichte Punktmenge.

Weitere Untersuchungen über diesen Gegenstand sind von den Herren HARDY, LITTLEWOOD [2]), WEYL [3]) und Anderen angestellt worden. Insbesondere haben die Herren HARDY und LITTLEWOOD in ihrer berühmten Acta-Abhandlung [2]) einige Fragen über die mögliche Ordnung der Annäherung aufgeworfen und zum Teile gelöst. Diese Frage-

[1]) Wobei, wenn nötig, die beiden extremen Fälle $\alpha_i = 0$ und $\alpha_i = 1$ nach Definition zu einem Fall zusammengefasst werden können.

[2]) G. H. HARDY and J. E. LITTLEWOOD, *Some problems of diophantine approximation,* [Acta Mathematica, t. XXXVII (1914), S. 155-190].

[3]) H. WEYL, *Über die Gleichverteilung von Zahlen mod. Eins* [Mathematische Annalen, Bd. 77 (1916), S. 313-352].

Khintchine's Palermo paper (1926): inhomogeneous result

- Khintchine For any θ there exists η such that

$$\inf_q q||q\theta - \eta|| > 0$$

- Tseng For any θ the set

$$\{\eta \in \mathbb{R} : \inf_q q||q\theta - \eta|| > 0\}$$

is winning

- I will skip some history

(Kleinbock, Kristensen, Bugeaud, Einsiedler, Tseng, Moshchevitin, Harrap, Bengoechea)

Erdős again

Primitive inhomogeneous approximation to one number

Chalk and Erdős (1959) Let $\theta \in \mathbb{R} \setminus \mathbb{Q}$ and $\eta \in \mathbb{R}$, there exists an absolute constant C such that

$$|q\theta - \eta - r| \leq \frac{C}{q} \cdot \left(\frac{\log q}{\log \log q} \right)^2$$

is satisfied by infinitely many coprime integers (q, r), $q \geq 1$.

Some modern history

- The result by Chalk and Erdős (1959) was forgotten
- Modern history: M. Laurent, A. Nogueira, A. Haynes ... Results weaker than theorem by Chalk and Erdős
- The result by Chalk and Erdős was rediscovered by Martin Widmer (2012)
- Laurent and Nogueira posed a problem to replace bound in RHS of the inequality by Chalk and Erdős by $\frac{C}{q}$
- Jitomirskaya, Liu (2019) For any constant C, there exists $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ and η such that the inequality

$$q\,|q\theta - \eta - r| \leq C$$

only has finitely many coprime integer solutions $(q, r), q \geq 1$.

(Non)primitive inhom. appr. to one number: new result

Theorem 1. (N.M.,Canadian Mathematical Bulletin (2022))
There exists $\theta \in \mathbb{R} \setminus \mathbb{Q}$ *and* η *such that*

$$\inf_{(q,r)\in\mathbb{Z}^2,\ q>100,\ (q,r)=1} q\, \frac{\log\log q}{\sqrt{\log q}}\, |q\theta - \eta - r| > 0.$$

Background:

- Erdős' Chinese Remainder result modified
- Inhomogeneous version of a theorem by Worley

Around Legendre Theorem

$$(p, q) = 1$$

- Legendre (18 century):

$$\left|\theta - \frac{p}{q}\right| < \frac{1}{2q^2}, \quad \Longrightarrow \quad \frac{p}{q} \text{ is a convergent } \frac{p_n}{q_n} \text{ to } \alpha$$

- Fatou (1904):

$$\left|\theta - \frac{p}{q}\right| < \frac{1}{q^2} \quad \Longrightarrow \quad \frac{p}{q} \text{ one of } \frac{p_n}{q_n}, \quad \frac{p_{n+1} + p_n}{q_{n+1} + q_n}, \quad \frac{p_{n+1} - p_n}{q_{n+1} - q_n}$$

- Worley (1981):

$$\left|\theta - \frac{p}{q}\right| < \frac{c}{q^2}, \quad c > 0 \quad \Longrightarrow \quad (p, q) \text{ is of the form}$$

$$(p, q) = (rp_{n+1} \pm sp_n, rq_{n+1} \pm sq_n) \text{ with } r, s \in \mathbb{Z}, \quad rs < 2c$$

Story No. 3

Inhomogeneous approximation
Kronecker's theorem in higher dimensions

A "new" multidimensional observation

$1, \theta_1, ..., \theta_n$ are linearly independent over $\mathbb{Q}$

- simultaneous approximation:

$$\forall \varepsilon > 0 \; \forall \boldsymbol{\theta} = (\theta_1, ..., \theta_n) \; \forall \boldsymbol{\eta} = (\eta_1, ..., \eta_n) \in \mathbb{R}^n$$

$$\exists \text{ inf. many primitive } (q, p_1, ..., p_n) \in \mathbb{Z}^{n+1}$$

$$\text{such that } \max_{j=1,..,n} |q\theta_j - p_j - \eta_j| < \varepsilon$$

- linear form $n \geq 2$:

$$\forall \varepsilon > 0 \; \forall \boldsymbol{\theta} = (\theta_1, ..., \theta_n) \; \forall \eta \in \mathbb{R}$$

$$\exists \text{ inf. many primitive } (q_0, q_1, .., q_n) \in \mathbb{Z}^{n+1}$$

$$\text{such that } |q_0 + \theta_1 q_1 + ... + \theta_n q_n - \eta| < \frac{\varepsilon}{\max_{1 \leq j \leq n} |q_j|}$$

Classical methods

- Studying successive minima of parallelepipeds

$$\Pi = \{\boldsymbol{z} \in \mathbb{R}^d : \ |\overline{\Theta}\boldsymbol{z}| < \varepsilon\}$$

- Mahler theory

- Constructing of a basis of the lattice $\mathbb{Z}^d$
(fundamental domain of the lattice $\mathbb{Z}^d$) in Π

- Shifting of fundamental domain

Kronecker's Theorem

approximations to solutions of equation $\Theta \boldsymbol{x} - \boldsymbol{\eta} + \boldsymbol{y} = \boldsymbol{0}$

$\Theta, \boldsymbol{\eta}$ -fixed, $\boldsymbol{x} \in \mathbb{Z}^m, \boldsymbol{y} \in \mathbb{Z}^n$

$$\Theta = \begin{pmatrix} \theta_{1,1} & \cdots & \theta_{1,m} \\ \theta_{2,1} & \cdots & \theta_{2,m} \\ \cdots & \cdots & \cdots \\ \theta_{n,1} & \cdots & \theta_{n,m} \end{pmatrix}, \overline{\Theta} = \begin{pmatrix} \theta_{1,1} & \cdots & \theta_{1,m} & -1 & \cdots & 0 \\ \theta_{2,1} & \cdots & \theta_{2,m} & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \theta_{n,1} & \cdots & \theta_{n,m} & 0 & \cdots & -1 \end{pmatrix}$$

$\psi_{\Theta^\top}(t) = \min\limits_{\boldsymbol{y} \in \mathbb{Z}^n : 0 < |\boldsymbol{y}| \le t} ||\Theta^\top \boldsymbol{y}||$ - irrationality measure function for $\Theta^\top$.

Theorem

Let $\boldsymbol{y}^\top \Theta \notin \mathbb{Z}^m$ for every $\boldsymbol{y} \in \mathbb{Z}^n \setminus \{\boldsymbol{0}\}$ (that is $\psi_{\Theta^\top}(t) \ne 0$).
Then for every $\varepsilon > 0$ and every $\boldsymbol{\eta} \in \mathbb{R}^n$ there exists an integer point $\boldsymbol{z} = \begin{pmatrix} \boldsymbol{x} \\ \boldsymbol{y} \end{pmatrix} \in \mathbb{Z}^{m+n}$ such that

$$|\overline{\Theta} \boldsymbol{z} - \boldsymbol{\eta}| = ||\Theta \boldsymbol{x} - \boldsymbol{\eta}|| < \varepsilon.$$

Khintchine's Palermo paper (1926): singularity

- uniform approximation
- singulatity phenomenon

$$\psi_\Theta(t) = \min_{|\boldsymbol{x}|\le t} ||\Theta \boldsymbol{x}|| - \text{ irrationality measure function}$$

$m, n = 1; \Theta = \theta$: $\limsup_{t\to\infty} t\psi_\theta(t) \ge \frac{1}{2} + \frac{1}{2\sqrt{5}}$ (Szekeres constant).

for example $m = 2, n = 1$: for any $\varphi \downarrow 0$ there exists $\Theta = (\theta_1, \theta_2)$ such that

$$0 < \psi_\Theta(t) = \min_{\boldsymbol{x}=(x_1,x_2):|\boldsymbol{x}1\le t} ||x_1\theta_1 + x_2\theta_2|| < \varphi(t).$$

another example $m = 1, n = 2$: for any $\varphi \downarrow 0$ with $\lim_{t\to\infty} t \cdot \varphi(t) = \infty$ there exists $\Theta = (\theta_1, \theta_2)^\top$ such that

$$0 < \psi_\Theta(t) = \min_{q\in\mathbb{Z}_+:q\le t} \max_{j=1,2} ||q\theta_j|| < \varphi(t).$$

singularity: Khintchine and Jarník

Jarník (1959): two cases

(1) $m \geq 2, n \geq 1$ for any $\varphi \downarrow 0$ there exists "many" Θ such that

$$\psi_\Theta(t) < \varphi(t)$$

(2) $m = 1, n \geq 2$ for any φ under the condition $t \cdot \varphi(t) \to \infty$ there exists "many" Θ such that

$$\psi_\Theta(t) < \varphi(t)$$

Transference principle: from singularity to inhomogeneous setting

(a) *Case* $n \geq 2$. *for any function* $\varphi(t)$ *with*

$$\lim_{t \to +\infty} \varphi(t) = 0$$

one can find a $m \times n$ *Matrix* Θ *and* $\boldsymbol{\eta} \in \mathbb{R}^n$ *such that*

1) $\theta_{i,j}, 1 \leq i \leq m,\ 1 \leq j \leq n$ *algebraically independent;*

2) *for inhomogeneous approximation one has*

$$||\Theta \mathbf{x} - \boldsymbol{\eta}|| \geq \varphi(|\mathbf{x}|), \quad \forall \mathbf{x} \in \mathbb{Z}^m \setminus \{\mathbf{0}\}. \tag{1}$$

(b) *Case* $n = 1$ *and* $m \geq 2$. *Let* $\varphi(t)$ *satisfy*

$$\lim_{t \to +\infty} t \cdot \varphi(t) = 0.$$

Then there exists $m \times 1$ *Matrix* $\Theta = (\theta_1, ..., \theta_m)$ *and* $\eta \in \mathbb{R}$ *such that*

1) $\theta_i, 1 \leq i \leq m$ *are algebraic independent*

2) *Inequality (1) is valid.*

Kronecker revisited: optimal result for $m \geq 2, n = 1$

$$\Theta = (\theta_1, ..., \theta_m).$$

We assume that there exist $1 \leq i < j \leq m$ such that $1, \theta_i, \theta_j$ are independent over $\mathbb{Q}$.

Then for any $\varepsilon > 0$ and any $\eta \in \mathbb{R}$ there exists $\boldsymbol{z} = \begin{pmatrix} \boldsymbol{x} \\ y \end{pmatrix} \in \mathbb{Z}^{m+1}$ with

$$|\overline{\Theta}\boldsymbol{z} - \boldsymbol{\alpha}| = ||\theta_1 x_1 + ... + \theta_m x_m - \eta|| < \frac{\varepsilon}{|\boldsymbol{x}|}.$$

Proof: relies on Jarník's Tbilissi observation (1938)

Coprime approximation again

Multidimensional metrical result: systems of linear forms

Dani, Laurent, Nogueira (2014):

$m, n \in \mathbb{Z}_+$; $\psi(x)$ such that $x \mapsto x^{m-1}\psi(x)^m$ non increasing

$$\sum_j j^{m-1}\psi(j)^n \quad \text{diverges} \quad / \quad \text{converges}$$

Then for almost all pairs

$$(\Theta, \boldsymbol{y}): \quad \Theta \in \mathrm{Mat}_{n,m}(\mathbb{R}), \quad \boldsymbol{y} \in \mathbb{R}^n$$

there exists infinitely many / only finitely many primitive points $(\boldsymbol{q}, \boldsymbol{p}) \in \mathbb{Z}^{m+n}$ $\boldsymbol{q} = (q_1, ..., q_m) \in \mathbb{Z}^m$, $\boldsymbol{p} = (p_1, ..., p_n) \in \mathbb{Z}^n$ with

$$|\Theta\boldsymbol{q} - \boldsymbol{p} - \boldsymbol{y}| \leq \psi(|\boldsymbol{q}|).$$

Story No. 4

Multidimensional and strange

$\mathrm{SL}_d(\mathbb{Z})$ approximations: ε-Kronecker

(i) $\varepsilon > 0$, $\psi_{\Theta^\top}(t) \neq 0$. Then there exist inf. many

$$\mathcal{Z} = \begin{pmatrix} z_1(1) & \dots & z_1(d) \\ \vdots & \vdots & \vdots \\ z_d(1) & \dots & z_d(d) \end{pmatrix} = (\boldsymbol{z}(1), \dots, \boldsymbol{z}(d)) \in \mathrm{SL}_d(\mathbb{Z})$$

with $\quad |\overline{\Theta}\boldsymbol{z}(\nu)| < \varepsilon, \;\; \nu = 1, \dots, d$

(ii) (N.M. 2016, conjectured by N. Chevallier) $n = 2, m = 1$
Let $\varphi(t) \downarrow 0$, $t \to \infty$. then there exists $(1, \theta_1, \theta_2) \in \mathbb{R}^3$ such that

1) $1, \theta_1, \theta_2$ linearly independent over $\mathbb{Q}$.

2) for any $\begin{pmatrix} x' & x'' & x''' \\ y_1' & y_1'' & y_1''' \\ y_2' & y_2'' & y_2''' \end{pmatrix} \in \mathrm{SL}_3(\mathbb{Z}), \quad x', x'', x''' \geq 1$ one has

$$\max\left\{ \frac{\max_{j=1,2} |x'\theta_j - y_j'|}{\varphi(x')}, \frac{\max_{j=1,2} |x''\theta_j - y_j''|}{\varphi(x'')}, \frac{\max_{j=1,2} |x'''\theta_j - y_j'''|}{\varphi(x''')} \right\} \geq 1.$$

$\mathrm{SL}_d(\mathbb{Z})$ approximations: one linear form

$n = 1, m \geq 2$

(i) Let. $1, \theta_1, ..., \theta_m$ linearly independent over $\mathbb{Q}$. Then for any $\varepsilon > 0$ there exists infinitely many matrices

$$\mathcal{Z} = (\boldsymbol{z}(1), ..., \boldsymbol{z}(d)) \in \mathrm{SL}_{n+1}(\mathbb{Z}),$$

$$\boldsymbol{z}(\nu) = (\boldsymbol{x}(\nu), y(\nu)), \ \ \boldsymbol{x}(\nu) \in \mathbb{Z}^m, \ \ y(\nu) \in \mathbb{Z} \qquad \text{with}$$

$$|\overline{\Theta}\boldsymbol{z}(\nu)| = ||\theta_1 x_1(\nu) + ... + \theta_m x_m(\nu)|| < \frac{\varepsilon}{|\boldsymbol{x}(\nu)|}, \ \ \nu = 1, ..., m+1$$

(ii) Conjecture. For any $m \geq 2$ and for any function $\varphi(t) \downarrow 0$ there exists a vector $\Theta = (\theta_1, ..., \theta_m)$ such that

1) $1, \theta_1, .., \theta_m$ linearly independent over $\mathbb{Q}$.

2) For any unimodular integer matrix $\mathcal{Z}$ one has

$$\max_{1 \leq \nu \leq m+1} \frac{|\boldsymbol{x}(\nu)| \cdot |\theta_1 x_1(\nu) + ... + \theta_m x_m(\nu) - y(\nu)|}{\varphi(|\boldsymbol{x}(\nu)|)} \geq 1$$

THANK YOU!